Problem Session

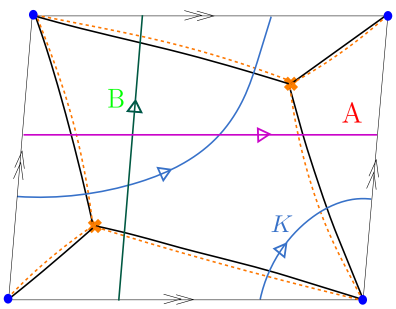
Geometrical structures in 4d $N = 2$ class S theories

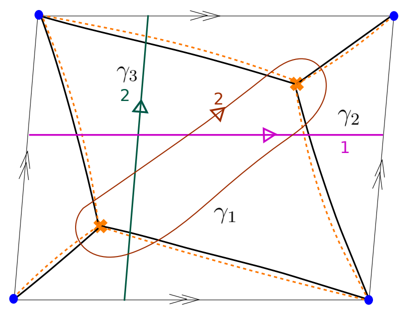
1 Line defects in the 4d $SU(2)$ $N = 2^*$ theory

Consider the 4d $SU(2)$ $N = 2^*$ theory, obtained from the 4d $SU(2)$ $N = 4$ super Yang-Mills by giving a mass to an adjoint hypermultiplet. The corresponding Riemann surface in its class S construction is a once-punctured torus, while the Seiberg-Witten curve is given by

$$\lambda^2 = (m^2\mathfrak{p}(z|\tau) + u)dz^2,$$

where $\mathfrak{p}(z|\tau)$ is the Weierstrass function.





The moduli space of flat $SL(2, \mathbb{C})$ connections on the once-punctured torus with a fixed conjugacy class of monodromy around the puncture is identified with the space of $SL(2, \mathbb{C})$ matrices A, B, M, up to simultaneous conjugation, with $ABA^{-1}B^{-1} = M$, where $\mathrm{Tr}(M) = \mu + \mu^{-1}$ is fixed. The algebra of holomorphic functions on such a moduli space is generated by

$$F_A := \mathrm{Tr}A, \;\; F_B := \mathrm{Tr}B, \;\; F_K := \mathrm{Tr}(AB). \tag{1.1}$$

Such functions come into physical life when we consider the 4d $N = 2^*$ theory on $\mathbb{R}^3 \times S^1$, where they are the VEVs of line defects (corresponding to the A-, B- and K-cycles) wrapped around the circle.

- Show that

$$F_A^2 + F_B^2 + F_K^2 - F_A F_B F_K = \mu + \mu^{-1} + 2 \tag{1.2}$$

(Hint: For 2×2 matrices X with $\det X = 1$ we have $X^2 - (\mathrm{Tr}X)X + 1 = 0$.)

• At a certain chamber on the Coulomb branch, the UV-IR map for line defects implies the following expansions of $F_{A,B,K}$ in terms of the Darboux coordinates $\mathcal{X}_\gamma$. (Representatives of the homology classes $\gamma_{1,2,3}$ are shown in the above figure.)

$$\begin{aligned} F_A &= \mathcal{X}_{\gamma_2} + \mathcal{X}_{-\gamma_2} + \mathcal{X}_{\gamma_1-\gamma_2}, \\ F_B &= \mathcal{X}_{\gamma_3} + \mathcal{X}_{-\gamma_3} + \mathcal{X}_{-\gamma_1+\gamma_3}, \\ F_K &= \mathcal{X}_{\gamma_2+\gamma_3} + \mathcal{X}_{-\gamma_2-\gamma_3} + \mathcal{X}_{\gamma_1-\gamma_2+\gamma_3} + \mathcal{X}_{-\gamma_1-\gamma_2+\gamma_3} + 2\mathcal{X}_{-\gamma_2+\gamma_3} \end{aligned} \tag{1.3}$$

Moreover here we also have $\mu = -\mathcal{X}_{\gamma_1-2\gamma_2-2\gamma_3}$.
Check that (1.2) holds for the expansions in (1.3).

2 Exact WKB for the Schrödinger equation

Consider the complex Schrödinger equation

$$\left[\hbar^2\partial_z^2 - V(z)\right]\psi(z,\hbar) = 0, \tag{2.1}$$

where $V(z)$ is holomorphic or meromorphic in z. Generically V also has nontrivial $\hbar$-dependence. Here we assume that V is $\hbar$-independent for simplicity.

• Given the WKB ansatz

$$\psi(z,\hbar) = \exp\left(\frac{1}{\hbar}\int_{z_0}^{z}\lambda(z',\hbar)dz'\right), \tag{2.2}$$

show that in order for $\psi(z,\hbar)$ to satisfy (2.1), λ has to obey the following Riccati equation

$$\lambda^2 - V + \hbar\partial_z\lambda = 0. \tag{2.3}$$

• Construct the formal solution to (2.3) as a formal series in $\hbar$:

$$\lambda^{\text{formal}} = \lambda^{(0)} + \sum_{n=1}^{\infty}\hbar^n\lambda^{(n)}. \tag{2.4}$$

Show that $\lambda^{(n)}$ is uniquely fixed once we choose a sheet of the following Riemann surface

$$\Sigma = \{\left(\lambda^{(0)}\right)^2 - V = 0\}. \tag{2.5}$$

Write down the first few terms in $\hbar$ of the formal solution. What can you say about $\lambda^{(\text{odd})}$? Note that substituting λ^{formal} back into the WKB ansatz (2.2) produces formal solution ψ^{formal} to the Schrödinger equation (2.1).

• Consider the Airy equation with $V(z) = z$. Show that one can construct two formal solutions

$$\psi_\pm^{\text{formal}}(z, \hbar) = \mathrm{e}^{\pm\hbar\frac{2}{3}z^{3/2}} \sum_{n=0}^{\infty} \psi_\pm^{(n)}(z)\hbar^n, \tag{2.6}$$

where $\psi_\pm^{(n)} \propto z^{-\frac{1}{4}-\frac{3}{2}n}$. The coefficients here are important, so it would be nice if you work them out.

• One way to resum an asymptotic series is the Borel resummation. Given

$$f(\hbar) \sim \mathrm{e}^{-\hbar^{-1}S_0} \sum_{n=0}^{\infty} c_n \hbar^n, \tag{2.7}$$

its Borel transform is

$$\mathcal{B}f(s) = \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(n)} (s - S_0)^{n-1}. \tag{2.8}$$

The Borel resummation is defined as the Laplace transformation of $\mathcal{B}f(s)$:

$$\mathcal{L}_\theta \left[\mathcal{B}f\right](\hbar) = \int_{S_0}^{\mathrm{e}^{\mathrm{i}\theta}\infty} ds \mathrm{e}^{-\frac{s}{\hbar}} \mathcal{B}f(s), \;\; \theta := \arg(\hbar) \tag{2.9}$$

We say that f is Borel summable if there are no singularities along the integration contour.

Based on the result of Kawai and Takei, in the neighborhood of a simple turning point z_0, one of the formal solutions $\psi_\pm^{\text{formal}}$ is not Borel summable when

$$\operatorname{Im}\left(\hbar^{-1}S_0(z)\right) = 0, \tag{2.10}$$

where

$$S_0(z) := \int_{z_0}^{z} \lambda^{(0)}(z')dz'. \tag{2.11}$$

The loci in the z-plane where (2.10) happens are denoted as Stokes lines. Draw the Stokes lines for the Airy equation.