Problems on QFT and stable envelopes

Mass in supersymmetric quantum mechanics (SQM)

Consider equivariant SQM with target space $X = \mathbb{C}$ (free chiral multiplet), on which $U(1)$ acts via a rotation $V = \frac{\partial}{\partial\varphi}$, the Morse function is $f = \frac{1}{2}m|z|^2$, where z is the standard coordinate on $\mathbb{C}$, and the metric is standard. That is, the supercharge is $\mathcal{Q} = e^{-f}(\mathrm{d}+\iota_{\epsilon V})e^f = \mathrm{d} + \mathrm{d}f + \iota_{\epsilon V}$, and the Hamiltonian is $H = \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\}$.

- Writing the L^2-ground state as $\Psi^{(m)} = e^{-f}\Omega^{(m)}$, where $\Omega^{(m)}$ is some equivariant differential form on X, find this $\Omega^{(m)}$.
- Using the SQM with time-dependent Morse function as described in lecture (in which the Hamiltonian is $H = \frac{1}{2}\{\mathcal{Q}, \mathcal{Q}^\dagger\} - i\frac{\partial f}{\partial t}$), consider $f = \frac{1}{2}m(t)|z|^2$, where the mass is m_1 for $t < 0$ and m_2 for $t > 0$, (i.e., it jumps by $m_2 - m_1$ at $t = 0$). This jump defines an operator acting on the space of states that we can refer to as the interface $\widehat{I}$. Compute the vacuum-vacuum transition amplitude $\langle \mathrm{vac}_{m_1}|\widehat{I}|\mathrm{vac}_{m_2}\rangle = R[m_1 \to m_2]$ of this interface. Check that it obeys

$$R[m_2 \to m_3]R[m_1 \to m_2] = R[m_1 \to m_3]. \tag{0.1}$$

- What is the limit of $\Omega^{(m)}$ as $m \to +\infty$ and for $m \to -\infty$?
- Do the same for a free hypermultiplet. That is, $X = \mathbb{C}^2$, $f = \frac{1}{2}m(|z_1|^2 - |z_2|^2)$, and we work equivariantly with respect to $U(1) \times U(1)$ acting on the two factors of $\mathbb{C}$ (with equivariant parameters ϵ_1 and ϵ_2). What is the $m \to \infty$ limit of $R[m \to -m]$ in this case?

Domain wall in SQED$_2$

Consider SQED$_2$, that is a gauge theory with eight supercharges with hypers valed in $\mathcal{R}\oplus\overline{\mathcal{R}}$, where $\mathcal{R} = \mathbb{C} \oplus \mathbb{C}$, both summands being the charge 1 representations of the gauge group $G = U(1)$ (so in $\overline{\mathcal{R}} = \mathbb{C} \oplus \mathbb{C}$ both $\mathbb{C}$'s have the charge -1). The real F.I. parameter is ζ, the real mass is m. Assuming that $e^2\zeta = 2m^2$ (here e is the gauge coupling), find the two fixed points (isolated massive vacua) on the Higgs branch $X_H = T^*\mathbb{C}P^1$ and solve analytically for the supersymmetric gradient flow trajectories that connect them. Such trajectories represent domain walls between the two massive vacua. As we send $m \to \infty$, what is the energy of such a domain wall (in quantum mechanics) or energy per unit volume (in higher-dimensional QFT)? (*Look up or ask for any undefined physical notions.*)

Motivation for elliptic cohomology

Let $\mathcal{X} = \mathrm{Map}(\mathbb{T}^2, \mathbb{C})$ be the appropriate space of complex functions on the two-torus (determine yourself what space should be considered). It is acted on by $S^1_1 \times S^1_2 \times S^1_3$, where the first two circles rotated the A and B cycles of the torus, and the third S^1 rotates $\mathbb{C}$ around the origin $0 \in \mathbb{C}$. Let us try to take the equivariant cohomology of $\mathcal{X}$,

$$H^\bullet_{S^1_1 \times S^1_2 \times S^1_3}(\mathcal{X}), \tag{0.2}$$

seriously. It is expected to appear as the description of supersymetric ground states of a free 3d chiral multiplet on $\mathbb{T}^2 \times \mathbb{R}$, viewed as an SQM with target $\mathcal{X}$. By applying the Atiah-Bott localization, we expect that every equivariant cohomology class localizes to the fixed point of $S^1_1 \times S^1_2 \times S^1_3$. What is this fixed point? Compute the Euler character of the normal bundle to the fixed point as a function of equivariant parameters (let us denote the equivariant parameters of $S^1_1 \times S^1_2 \times S^1_3$ by β, τ and a, respectively). Observe that it is not a function, but rather a section: Of what bundle and on what base? *Hint: you might need to use the zeta-function or some other regularization.*

Chern-Simons levels and factors of automorphy

Consider a 3d QFT that has a unique gapped vacuum $|0\rangle$. Let this vacuum preserve global symmetry $U(1)^r$. Coupling to the background gauge fields for this symmetry (the vacuum response to the external magnetic field) is described, in the leading order, by the effective Chren-Simons action

$$\Gamma[A] = \frac{k_{ij}}{4\pi} \int A^i \mathrm{d}A^j, \tag{0.3}$$

where k_{ij} is some symmetric matrix of integers. Consider this theory on $\mathbb{E}_\tau \times \mathbb{R}$, where we turn on flat connections for the $U(1)^r$ along the torus $\mathbb{E}_\tau$. Such flat connections are parameterized by $\mathcal{E} \cong (\mathbb{E}^\vee_\tau)^r$, and the vacuum $\mathbb{C}|0\rangle$ fibers non-trivially over $\mathcal{E}$. Determine the natural holomorphy structure induced by the Berry connection, as well as the factor of automorphy of this line bundle.