Crossing Beyond Scattering Amplitudes

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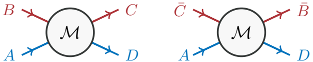
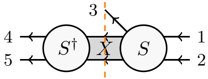
Outline

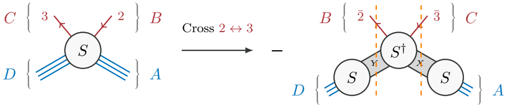
1. Introduction



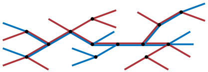
2. What can be measured asymptotically?

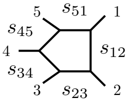


3. Crossing equation



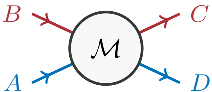
4. Examples

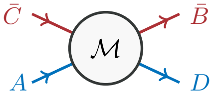




Review on crossing symmetry

Amplitudes for $AB \to CD$ and $A\bar{C} \to \bar{B}D$ are boundary values of the **same analytic function**





$$\mathcal{M}_{AB\to CD} \quad \underset{\substack{\text{Analytic}\\ \text{continuation}}}{\longleftrightarrow} \quad \mathcal{M}_{A\bar{C}\to\bar{B}D}$$

Particles indistinguishable from antiparticles traveling back in time?

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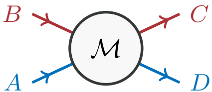
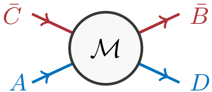




Not relabeling or cyclic invariance

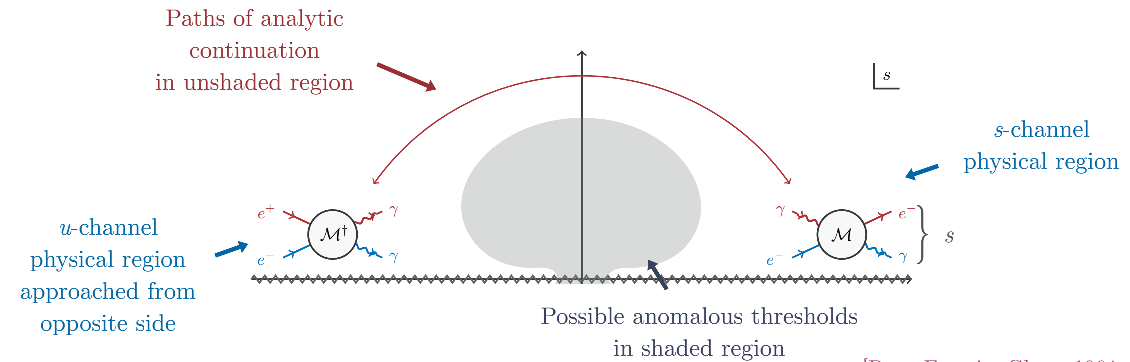
$$\mathcal{M}_{AB\to CD} \underset{\text{Analytic continuation}}{\longleftrightarrow} \mathcal{M}_{A\bar{C}\to\bar{B}D}$$

Particles indistinguishable from antiparticles traveling back in time?

Crossing symmetry would allow us to use results from previous computations:

Crossing symmetry in 2 to 2 scattering

Proven for the non-perturbative amplitude at fixed momentum transfer $t<0$
in theories with mass gap



[Bros, Epstein, Glaser 1964, 1965]

Crossing symmetry in 2 to 2 scattering

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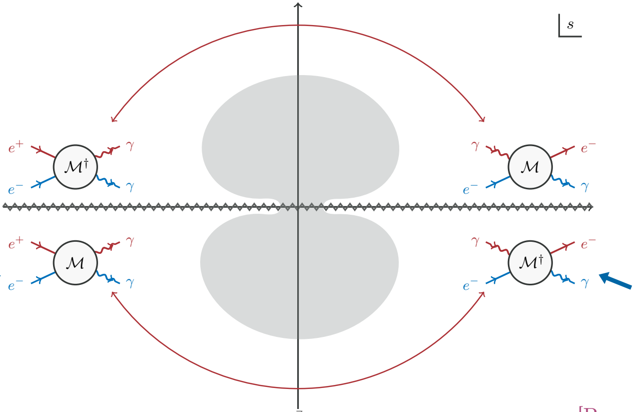


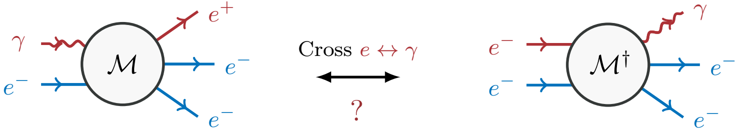
u-channel physical region

s-channel physical region approached from opposite side

[Bros, Epstein, Glaser 1964, 1965]

Crossing symmetry in 2 to 3 scattering

Same for the 5 pt amplitude, right?



CROSSING SYMMETRY IN 2 TO 3 SCATTERING

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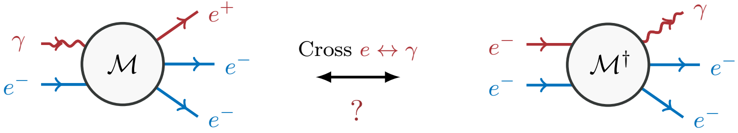
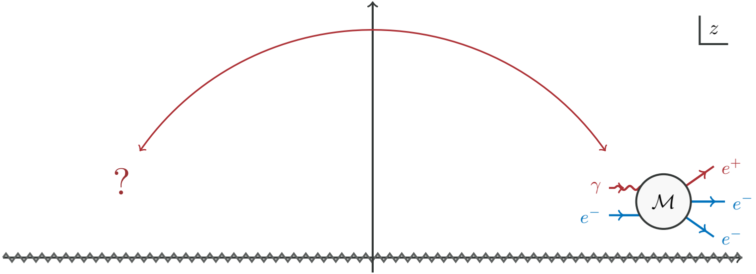


No.

Crossing symmetry in 2 to 3 scattering

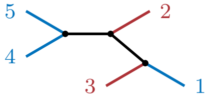
The central topic of this talk:

What is the result of analytically continuing scattering amplitudes?



Simple example at Tree level

$$\mathcal{M}_{543\leftarrow 21} =$$



$$= \frac{g^3}{(s_{45} - m_{45}^2 + i\varepsilon)(s_{13} - m_{13}^2)}\,,$$

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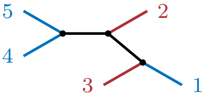

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Distributions at tree level useful for understanding loop level

We will take time flowing to the left!

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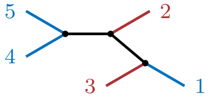
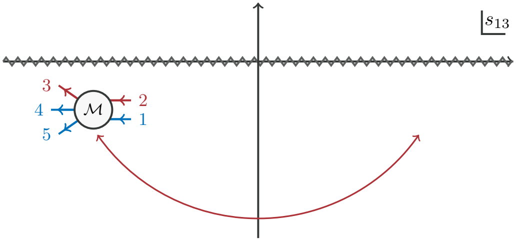


$$= \frac{g^3}{(s_{45} - m_{45}^2 + i\varepsilon)(s_{13} - m_{13}^2)}\,,$$

Rotate s_{13} in the lower half plane at fixed s_{45}



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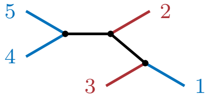

$$= \frac{g^3}{(s_{45}-m_{45}^2+i\varepsilon)(s_{13}-m_{13}^2)}\,,$$

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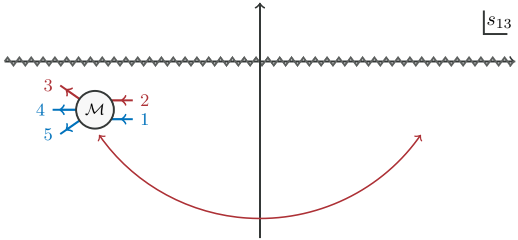
Exchanges
2 and 3



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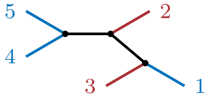
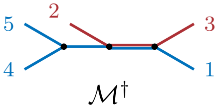
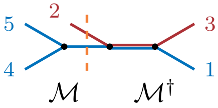


$$= \frac{g^3}{(s_{45} - m_{45}^2 + i\varepsilon)(s_{13} - m_{13}^2)}\,,$$

Rotate s_{13} in the lower half plane at fixed s_{45}

$$\begin{aligned}[\mathcal{M}_{543\leftarrow 21}]_{s_{13}\curvearrowright} &= \frac{g^3}{(s_{45} - m_{45}^2 + i\varepsilon)(s_{13} - m_{13}^2 - i\varepsilon)} \\ &= \underbrace{\frac{g^3}{(s_{45} - m_{45}^2 - i\varepsilon)(s_{13} - m_{13}^2 - i\varepsilon)}} - \underbrace{2\pi i \delta(s_{45} - m_{45}^2)\frac{g^3}{(s_{13} - m_{13}^2 - i\varepsilon)}}\end{aligned}$$



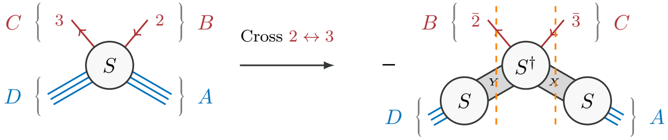


Takeaway point:

Analytic continuation *from* $\mathcal{M}$ *lands on something* ***new***

HERE: RELATE ASYMPTOTIC OBSERVABLES

We will learn: Scattering amplitudes are part of a **larger family of observables**, related by analytic continuation



Crossing equation describes the result of analytic continuation

Previous progress on crossing

- Proposed for quantum field theory in 1954

[Gell-Mann, Goldberger, Thirring]

- Proven for non-perturbative 2→2 and 2→3 scalar amplitudes, assuming mass gap
  - *Proofs use **mass gap, causality, unitarity,** and **analytic extension** theorems*

[Bros, Epstein, Glaser 1964, 1965; Bros 1986]

- Recent progress in Chern-Simons theories and string theory for 2→2 amplitudes

[See e.g. Jain, Mandlik, Minwalla, Takimi, Wadia, Yokoyama 2014; Lacroix, Erbin, Sen 2018; Mehta, Minwalla, Patel, Prakash, Sharma 2022; Gabai, Sandor, Yin 2022]

- Proven in the planar limit to any multiplicity using perturbation theory

[Mizera 2021]

Challenge: understand connection between crossing and physical principles

Outline

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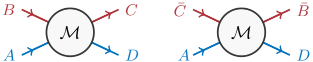

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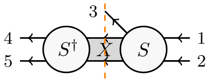

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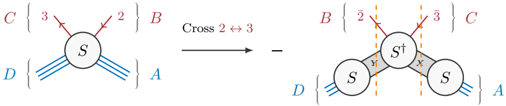

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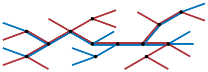

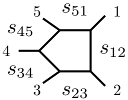

Asymptotic algebra in quantum field theory

1. Algebra of asymptotic measurements in the far past and far future,

a_{in}

$$\left[a_1, a_2^\dagger\right] = \delta_{1,2}\, 2p_1^0 (2\pi)^{\mathrm{D}-1} \delta^{\mathrm{D}-1}(\vec{p}_1 - \vec{p}_2)$$

a_{out}

$$\left[b_1, b_2^\dagger\right] = \delta_{1,2}\, 2p_1^0 (2\pi)^{\mathrm{D}-1} \delta^{\mathrm{D}-1}(\vec{p}_1 - \vec{p}_2)$$

2. These operators act on equivalent Hilbert spaces and are related by a unitary evolution operator S:

$$b = S^\dagger a S, \quad b^\dagger = S^\dagger a^\dagger S; \qquad SS^\dagger = \mathbb{1}$$

3. There exists a time-invariant vacuum $|0\rangle$:

$$a_i|0\rangle = b_i|0\rangle = 0, \quad S|0\rangle = |0\rangle$$

4. Stability:

$$Sa_i^\dagger|0\rangle = a_i^\dagger|0\rangle, \qquad Sb_i^\dagger|0\rangle = b_i^\dagger|0\rangle$$

Asymptotic algebra in quantum field theory

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Assume Bose/Fermi statistics, flat space, Poincaré invariance

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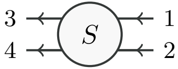
Using this algebra,

What can be measured asymptotically?

4 pt asymptotic measurements

$$\langle 0|b_4 b_3 a_2^\dagger a_1^\dagger|0\rangle = {}_{\rm in}\langle 43|S|21\rangle_{\rm in} =$$



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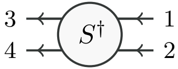

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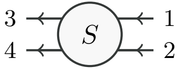

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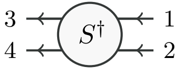

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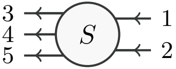
Time flows to the left in all diagrams

5 PT ASYMPTOTIC MEASUREMENTS

$$\langle 0 | b_5 b_4 b_3 a_2^\dagger a_1^\dagger | 0 \rangle = {}_{\text{in}}\langle 543 | S | 21 \rangle_{\text{in}} =$$



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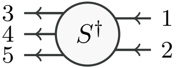

(plus forward terms and Hermitian conjugates)

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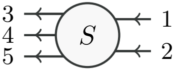

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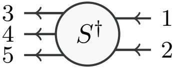

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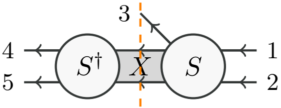

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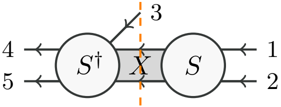

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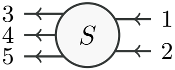
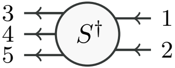


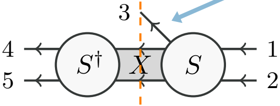
$$\langle 0|a_5 a_4 a_3 b_2^\dagger b_1^\dagger|0\rangle = {}_{\text{in}}\langle 543|S^\dagger|21\rangle_{\text{in}} =$$



Insert a complete set of states X, integrate inclusively over phase space

$b = S^\dagger a S$

$$\langle 0|a_5 a_4 b_3 a_2^\dagger a_1^\dagger|0\rangle = {}_{\text{in}}\langle 54|b_3|21\rangle_{\text{in}} =$$



$b^\dagger = S^\dagger a^\dagger S$

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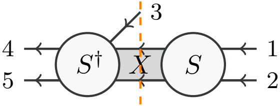
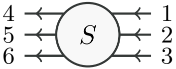


(plus forward terms and Hermitian conjugates)

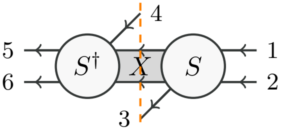
Example 6 pt asymptotic measurements

$\langle 0|b_6 b_5 b_4 a_3^\dagger a_2^\dagger a_1^\dagger|0\rangle$



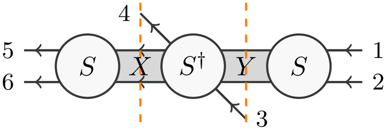
Scattering amplitudes

$\langle 0|a_6 a_5 b_4^\dagger b_3 a_2^\dagger a_1^\dagger|0\rangle$



Inclusive amplitudes

$\langle 0|b_6 b_5 a_4 b_3^\dagger a_2^\dagger a_1^\dagger|0\rangle$



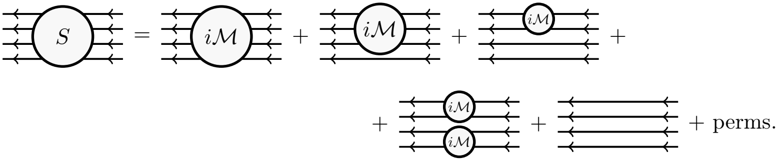
Out-of-time-ordered correlators

Example n-pt asymptotic measurements

More generally:

$$\langle 0|\underbrace{a\cdots a}_{k_{2s}}S\underbrace{a^\dagger\cdots a^\dagger}_{k_{2s-1}}\underbrace{a\cdots a}_{k_{2s-2}}S^\dagger\cdots S^\dagger\underbrace{a^\dagger\cdots a^\dagger}_{k_3}\underbrace{a\cdots a}_{k_2}S\underbrace{a^\dagger\cdots a^\dagger}_{k_1}|0\rangle$$

We expand in terms of connected components:



Physical interpretation of asymptotic observables

$${}_{\text{in}}\langle 0|b_n \cdots b_{j+1} a_j^\dagger \cdots a_1^\dagger|0\rangle_{\text{in}}$$: Scattering amplitude

$${}_{\text{in}}\langle 54|b_3|21\rangle_{\text{in}}$$: Expectation value of electromagnetic field in a scattering experiment / Gravitational waveform detected by LIGO-Virgo-KAGRA

$$\lim_{p_3 \to p_4} {}_{\text{in}}\langle 65|b_4 b_3^\dagger|21\rangle_{\text{in}}$$: Inclusive cross section / inclusive particle number

$${}_{\text{in}}\langle 6|b_5^\dagger a_4 b_3^\dagger a_2^\dagger|1\rangle_{\text{in}}$$: Out-of-time-ordered correlator

[See e.g. Shenker, Stanford 2013; Maldacena, Shenker, Stanford 2015; Kosower, Maybee, O'Connell 2018; Caron-Huot 2022]

Takeaway points:

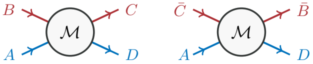
- S-matrix only one of exponentially many asymptotic observables
- Asymptotic observables are physical; already being measured and computed

In this talk:

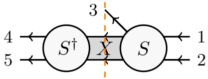
- Relate asymptotic observables to one another via analytic continuations

Outline

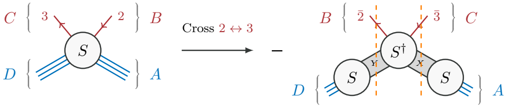
1. Introduction



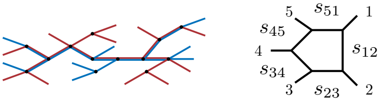
2. What can be measured asymptotically?



3. Crossing equation



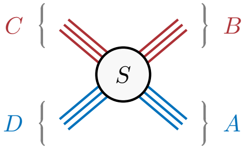
4. Examples

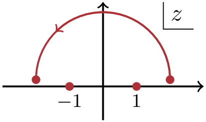


Contour of continuation

We propose an **on-shell** contour of analytic continuation which exchanges **incoming and outgoing** states with a parameter z

$$p_b^\mu(z) = \left(z p_b^+,\, \tfrac{1}{z} p_b^-,\, p_b^\perp\right), \quad \text{for all } b \in B \qquad p_c^\mu(z) = \left(z p_c^+,\, \tfrac{1}{z} p_c^-,\, p_c^\perp\right), \quad \text{for all } c \in C$$

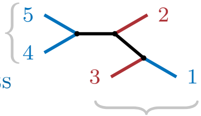




Crossing path in practice

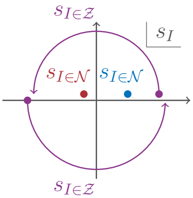
s_{45} stays fixed

Uniform invariants stay fixed

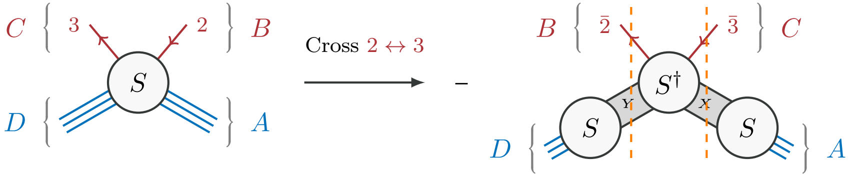


s_{13} rotates

Mixed invariants rotate

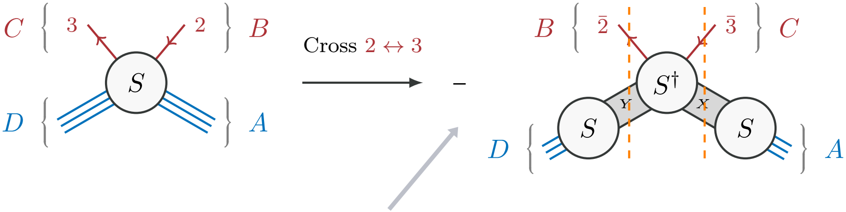


Crossing Equation for 2-particle crossing:



[See also Bros 1986]

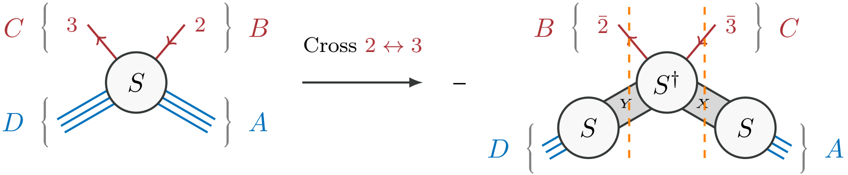
Crossing Equation for 2-particle crossing:



Minus sign from $S = \mathbb{1} + i(2\pi)^{\mathrm{D}}\delta^{\mathrm{D}}(\Sigma p_i)\mathcal{M}$

[See also Bros 1986]

Crossing Equation for 2-particle crossing:



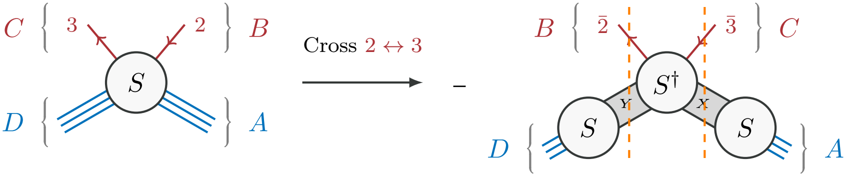
Evidence:

- Loop-level examples and tree-level proof (*part 4*)
- Axiomatic quantum field theory, assuming analyticity, using microcausality

$$[b, a^\dagger] \quad \underset{\text{Cross } B\leftrightarrow C}{\longleftrightarrow} \quad [b^\dagger, a]$$

[See also Bros 1986]

Crossing Equation for 2-particle crossing:



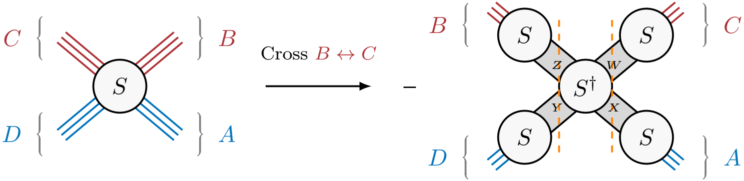
$$\mathcal{G}_{AB\to CD} - \mathcal{G}_{AC\to BD} = \int \mathrm{d}^{\mathrm{D}}x\, \mathrm{e}^{i(p_c - p_b)\cdot x} \langle D|[j(x/2), j(-x/2)]|A\rangle$$

Use **microcausality** $\mathcal{G}_{AB\to CD} - \mathcal{G}_{AC\to BD} = 0$:

$$[b, a^\dagger] \quad \overset{\longleftrightarrow}{\text{Cross } B\leftrightarrow C} \quad [b^\dagger, a]$$

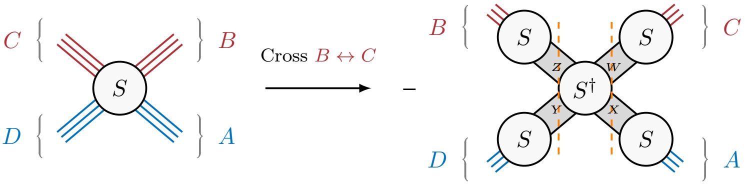
[Bros, Epstein, Glaser 1964, 1965; Itzykson, Zuber 1987]

Crossing proposal for multi-particle crossing:



$$[S_{DC\leftarrow BA}]_{\circlearrowright s_I, \curvearrowleft s_J} = \sum\!\!\!\!\!\!\!\int_{X,Y,Z,W} S_{D\leftarrow Z} S_{B\leftarrow Y} S^{\dagger}_{YZ\leftarrow XW} S_{W\leftarrow C} S_{X\leftarrow A}\,.$$

Crossing proposal for multi-particle crossing:



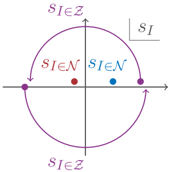
$$[S_{DC\leftarrow BA}]_{\circlearrowright s_I, \curvearrowleft s_J} = \sum_{X,Y,Z,W} S_{D\leftarrow Z} S_{B\leftarrow Y} S^\dagger_{YZ\leftarrow XW} S_{W\leftarrow C} S_{X\leftarrow A} \,.$$

Evidence:

- Loop-level examples and tree-level proof (*part 4*)
- Symmetry in AD & BC

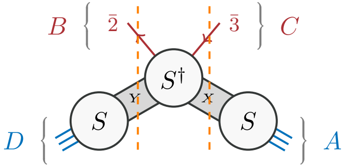
Proving the crossing equation involves comparing:

(I)



The analytic continuation of S via the prescribed path

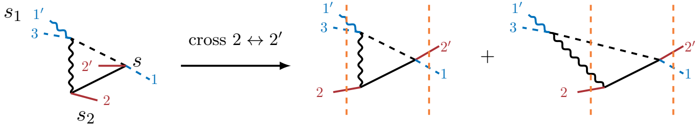
(II)



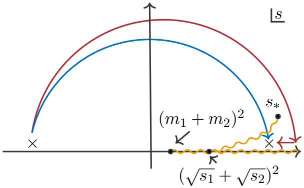
Computing the corresponding observables explicitly

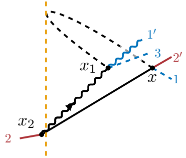
Continuing around singularities

Local analyticity can be subtle: might need to continue past **anomalous thresholds**



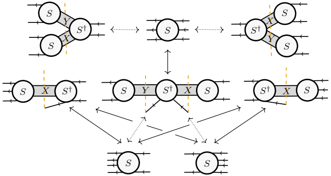
Expected from axiomatic field theory



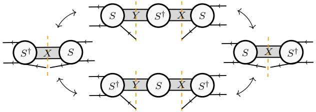


Families of observables

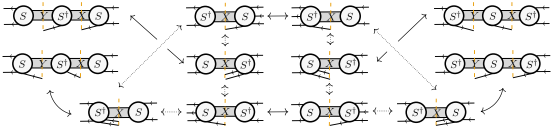
1



2

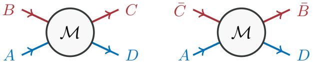


3

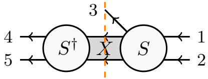


Outline

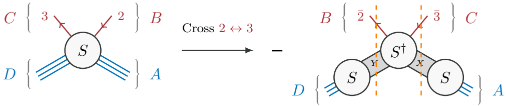
1. Introduction



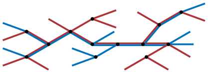
2. What can be measured asymptotically?

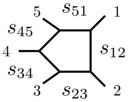


3. Crossing equation



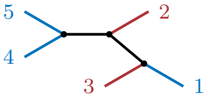
4. Examples





Tree-Level example revisited

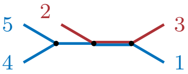
$$\mathcal{M}_{543\leftarrow 21} =$$

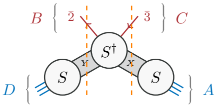


$$= \frac{g^3}{(s_{45} - m_{45}^2 + i\varepsilon)(s_{13} - m_{13}^2)}$$

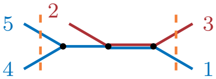
(I) Analytic continuation path: s_{13} rotates, s_{45} stays fixed,

$$[\mathcal{M}_{543\leftarrow 21}]_{s_{13}\curvearrowright} = \frac{g^3}{(s_{45} - m_{45}^2 + i\varepsilon)(s_{13} - m_{13}^2 - i\varepsilon)}$$

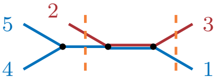
(II) Crossing prediction: all ways of fitting 

into 

Allowed patterns:

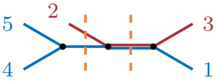


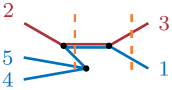
$$= \frac{g^3}{(s_{45} - m_{45}^2 - i\varepsilon)(s_{13} - m_{13}^2 - i\varepsilon)}$$

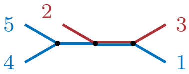


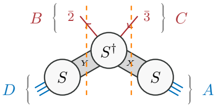
$$= -2\pi i \delta(s_{45} - m_{45}^2) \frac{g^3}{(s_{13} - m_{13}^2 - i\varepsilon)}$$

Example disallowed patterns:

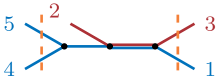




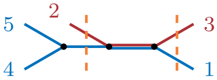
(II) Crossing prediction: all ways of fitting 

into 

Allowed patterns:



$$= \frac{g^3}{(s_{45} - m_{45}^2 - i\varepsilon)(s_{13} - m_{13}^2 - i\varepsilon)}$$

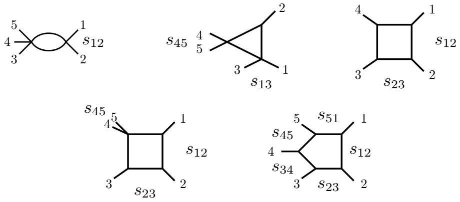


$$= -2\pi i \delta(s_{45} - m_{45}^2) \frac{g^3}{(s_{13} - m_{13}^2 - i\varepsilon)}$$

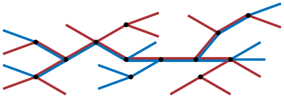
Comparing **(I)** and **(II)** verifies the crossing equation.

Perturbation theory checks

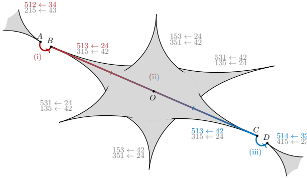
- Checked all D-dim massless basis integrals for an expansion around D=4

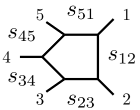


- Proof at any multiplicity at tree level (highly nontrivial)



Crossing check for pentagon





$$\left[\boldsymbol{I}_0^{(34\to215)}\right]_{2\leftrightarrow3} - \left[\boldsymbol{I}_0^{(24\to315)}\right]^* + \mathrm{Cut}_{s_{51}}\boldsymbol{I}_0^{(34\to215)} \overset{?}{=} 0$$

$$\left[\boldsymbol{I}_0^{(34\to215)}\right]_{2\leftrightarrow3} = \mathcal{P}\exp\left(\epsilon\int_{\gamma_{2\leftrightarrow3}} \mathrm{d}\boldsymbol{\Omega}\right)\cdot \boldsymbol{I}_0^{(34\to215)}$$

$$\left[\boldsymbol{I}_0^{(34\to215)}\right]_{2\leftrightarrow3} = \begin{pmatrix} -1 & -i\pi & \frac{7\pi^2}{12} & \frac{7\zeta_3}{3}+\frac{i\pi^3}{4} & -\frac{73\pi^4}{1440}+\frac{7i\pi\zeta_3}{3} \\ -\mathbf{1}_4 & \mathbf{0}_4 & \frac{\pi^2}{12}\mathbf{1}_4 & \frac{7\zeta_3}{3}\mathbf{1}_4 & \frac{47\pi^4}{1440}\mathbf{1}_4 \\ 2 & 2i\pi & -\frac{5\pi^2}{6} & -\frac{14\zeta_3}{3}-\frac{i\pi^3}{6} & -\frac{13\pi^4}{240}-\frac{14i\pi\zeta_3}{3} \\ 2 & 2i\pi & -\frac{7\pi^2}{6} & -\frac{20\zeta_3}{3}-\frac{i\pi^3}{6} & -\frac{7\pi^4}{144}-\frac{14i\pi\zeta_3}{3} \\ 2 & 0 & -\frac{\pi^2}{2} & -\frac{20\zeta_3}{3} & -\frac{43\pi^4}{720} \\ 2 & -2i\pi & -\frac{\pi^2}{2} & -\frac{20\zeta_3}{3}+\frac{i\pi^3}{6} & -\frac{43\pi^4}{720}+\frac{14i\pi\zeta_3}{3} \\ 2 & 0 & -\frac{\pi^2}{2} & -\frac{20\zeta_3}{3} & -\frac{43\pi^4}{720} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \epsilon \\ \epsilon^2 \\ \epsilon^3 \\ \epsilon^4 \end{pmatrix}.$$

$$\left[\boldsymbol{I}_0^{(24\to315)}\right]^* = \begin{pmatrix} -1 & i\pi & \frac{7\pi^2}{12} & \frac{7\zeta_3}{3}-\frac{i\pi^3}{4} & -\frac{73\pi^4}{1440}-\frac{7i\pi\zeta_3}{3} \\ -\mathbf{1}_4 & \mathbf{0}_4 & \frac{\pi^2}{12}\mathbf{1}_4 & \frac{7\zeta_3}{3}\mathbf{1}_4 & \frac{47\pi^4}{1440}\mathbf{1}_4 \\ 2 & -2i\pi & -\frac{5\pi^2}{6} & -\frac{14\zeta_3}{3}+\frac{i\pi^3}{6} & -\frac{13\pi^4}{240}+\frac{14i\pi\zeta_3}{3} \\ 2 & -2i\pi & -\frac{7\pi^2}{6} & -\frac{20\zeta_3}{3}+\frac{i\pi^3}{6} & -\frac{7\pi^4}{144}+\frac{14i\pi\zeta_3}{3} \\ 2 & 0 & -\frac{\pi^2}{2} & -\frac{20\zeta_3}{3} & -\frac{43\pi^4}{720} \\ 2 & 2i\pi & -\frac{\pi^2}{2} & -\frac{20\zeta_3}{3}-\frac{i\pi^3}{6} & -\frac{43\pi^4}{720}-\frac{14i\pi\zeta_3}{3} \\ 2 & 0 & -\frac{\pi^2}{2} & -\frac{20\zeta_3}{3} & -\frac{43\pi^4}{720} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \epsilon \\ \epsilon^2 \\ \epsilon^3 \\ \epsilon^4 \end{pmatrix}$$

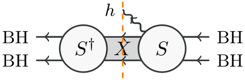
$$\mathrm{Cut}_{s_{51}}\boldsymbol{I}_0^{(34\to215)} = \begin{pmatrix} 0 & 2i\pi & 0 & -\frac{i\pi^3}{2} & -\frac{14}{3}i\pi\zeta_3 \\ \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{0}_4 & \mathbf{0}_4 \\ 0 & -4i\pi & 0 & \frac{i\pi^3}{3} & \frac{28i\pi\zeta_3}{3} \\ 0 & -4i\pi & 0 & \frac{i\pi^3}{3} & \frac{28i\pi\zeta_3}{3} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 4i\pi & 0 & -\frac{i\pi^3}{3} & -\frac{28}{3}i\pi\zeta_3 \\ \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 & \mathbf{0}_2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ \epsilon \\ \epsilon^2 \\ \epsilon^3 \\ \epsilon^4 \end{pmatrix}$$

[Chicherin, Henn, Mitev 2017]

Emission in black-hole scattering

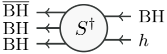
Waveform in LIGO-Virgo-KAGRA obtained as an in-in expectation value

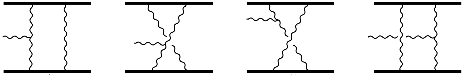
$${}_{\text{in}}\langle 54|b_3|12\rangle_{\text{in}} \quad =$$



[Kosower, Maybee, O'Connell 2018]

Here, **analytically continue** the 5-pt amplitude in one-loop computations





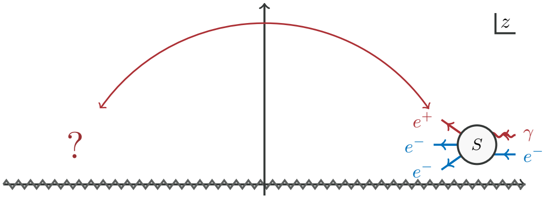
[See also Brandhuber, Brown, Chen, De Angelis, Gowdy, Travaglini 2023; Herderschee, Roiban, Teng 2023; Elkhidir, O'Connell, Sergola, Vazquez-Holm 2023]

Conclusions

- Exponentially many **asymptotic observables**, e.g. gravitational waveforms, out-of-time-ordered correlators and in-in expectation values

- **New version of crossing symmetry:**

S-matrix contains a host of asymptotic observables which are related by analytic continuations between different channels

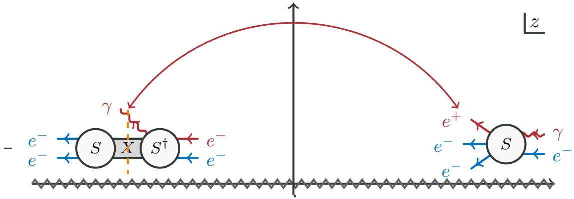


Conclusions

- Exponentially many **asymptotic observables**, e.g. gravitational waveforms, out-of-time-ordered correlators and in-in expectation values

- **New version of crossing symmetry:**

S-matrix contains a host of asymptotic observables which are related by analytic continuations between different channels



THANKS!