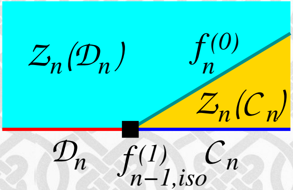
Symmetry/Topological-Order (Symm/TO) correspondence

Xiao-Gang Wen (MIT)

From string theory to condensed matter physics

2023/07, Strings 2023, PI

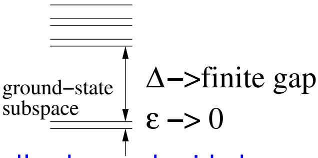
Kong Wen Zheng arXiv:1502.01690
Ji Wen arXiv:1905.13279
Ji Wen arXiv:1912.13492
Kong Lan Wen Zhang Zheng arXiv:2005.14178



Simons Collaboration on
Ultra-Quantum Matter

Three kinds of quantum phases

All quantum systems discussed here have **lattice UV completion** which defines **condensed matter systems**



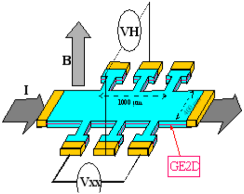
- **Gapped** → no low energy excitations
  All excitations has energy gap.
  Band insulators, FQH states
  General theory: topological order, moduli bundle theory, braided fusion higher category
- **Gapless (finite)** → finite low energy modes
  Finite low energy modes: Dirac/Weyl semimetal, superfluid, critical point at continuous phase transition
  General theory: quantum field theory, conformal field theory, ???
- **Gapless (infinite)** → infinite low energy modes
  Infinite low energy modes: Fermi metal, Bose metal, *etc*
  (Low energy effective theory is beyond quantum field theory)
  General theory: Landau Fermi liquid, ???

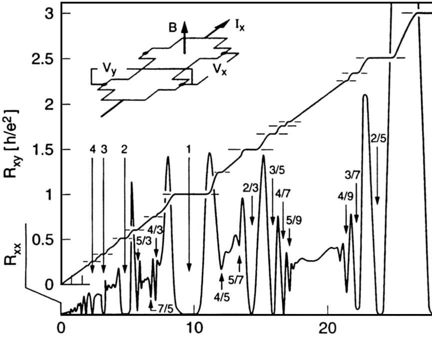
Topological orders in quantum Hall effect

For a long time, we thought that Landau symmetry breaking classify all phases of matter

- Quantum Hall states $R_{xy} = V_y/I_x = \frac{m}{n}\frac{2\pi\hbar}{e^2}$

von Klitzing Dorda Pepper, PRL **45** 494 (1980)
Tsui Stormer Gossard, PRL **48** 1559 (1982)



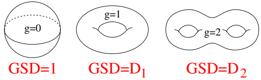


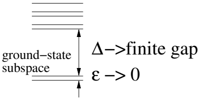
- FQH states have different phases even when there is no symm. and no symm. breaking.

- FQH liquids must contain a new kind of order, named as **topological order**

Characterize topological order quantitatively

- How to extract universal numbers (topological invariants) from complicated many-body wavefunction $\Psi(x_1, \cdots, x_{10^{20}})$





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[4]X. G. Wen, F. Wilczek, and A. Zee, Phys. Rev. B **39**, 11 413 (1989); D. Khveshchenko and P. Wiegmann (unpublished).

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Put the gapped system on space with various topologies, and measure the ground state degeneracy → topological order

Vacuum degeneracy of chiral spin states in compactified space

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(Received 10 May 1989)

A chiral spin state is not only characterized by the T and P order parameter $E_{123} = \mathbf{S}_1 \cdot (\mathbf{S}_2 \times \mathbf{S}_3)$, it is also characterized by an integer k. In this paper we show that this integer k can be determined from the vacuum degeneracy of the chiral spin state on compactified spaces. On a Riemann surface with genus g the vacuum degeneracy of the chiral spin state is found to be $2k^g$. Among those vacuum states, some k^g states have $\langle E_{123} \rangle > 0$, while other k^g states have $\langle E_{123} \rangle < 0$. The dependence of the vacuum degeneracy on the topology of the space reflects some sort of topological ordering in the chiral spin state. In general, the topological ordering in a system is classified by topological theories.

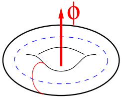
Ground state degen. characterizes phase of matter

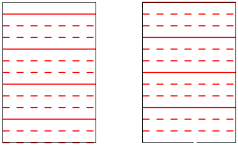
Objection: GSD on $S^2 \neq$ GSD on T^2 (coming from the motion of center mass). Ground state degeneracy is just a finite size effect. Ground state degeneracy does not reflect the thermodynamic phase of matter.

- **Robust topological ground state degeneracy**

- Inserting 2π flux pumps one quantum Hall ground state in magnetic field B to another ground state.

- k_x of the two ground states differ by $\Delta k_x \sim BL_y \to \infty|_{L_y\to\infty}$



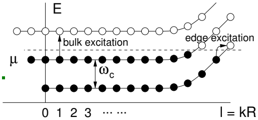


- Impurities can only cause momentum transfer $\delta k_x \sim \sqrt{B}$, and split ground state degeneracy by $\Delta E \sim \mathrm{e}^{-\# L_y \sqrt{B}}$

Wen Niu PRB 41, 9377 (90)

- Magnetic field $B \to$ UV-IR mixing and non-commutative geometry

Even non-Abelian statistics can be realized



Let $\chi_n(z_i)$ be the many-body wave function of n filled Landau level, which describes a gapped state.

- Products of gapped IQH wave functions χ_n are also gapped $\to$ new FQH states — Jain PRB **11** 7635 (90)
- $SU(m)_n$ state $\chi_1^k \chi_n^m$ via slave-particle — Wen PRL **66** 802 (1991)

$$\Psi_{SU(3)_2} = (\chi_2)^3, \ \nu = 2/3; \qquad \Psi_{SU(2)_2} = \chi_1(\chi_2)^2, \ \nu = 1/2;$$

$\to$ Effective $SU(3)_2$, $SU(2)_2$ Chern-Simons theory
$\to$ non-Abelian statistics (assume $\chi_1^k \chi_n^m$ is gapped, conjecture)

- Pfaffien state via CFT correlation — Moore-Read NPB **360** 362 (1991)

$$\Psi_{\text{Pfa}} = \mathcal{A}[\frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4}\cdots]\prod(z_i - z_j)^2 \mathrm{e}^{-\frac{1}{4}\sum |z_i|^2}, \qquad \nu = 1/2$$

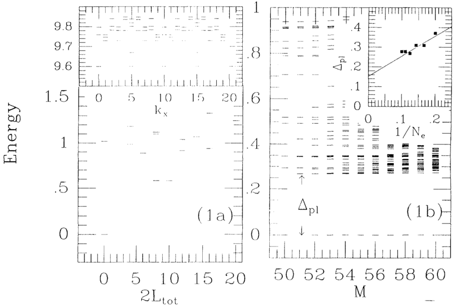
Conformal block $=$ multi-valueness of many-body wave function
$\overset{\text{conjecture}}{\to}$ non-Abelian Berry phase $\to$ non-Abelian statistics

Numerical confirmation of non-Abelian statistics

Application of TQFT/CFT correspondence. Witten, CMP **121** 352 (89)

- Edge state of Abelian FQH state (classified by K-matrices) always has an integral central charge $c \in \mathbb{N}$, Wen Zee PRB **46** 2290 (92)
- If edge states are described by a fractional central charge → The bulk must be a non-Abelian state. Wen PRL **70** 355 (93)
- For $\nu = 1/2$ state with a three-body interaction, the edge spectrum is given by
  (for 8 electrons on 20 orbits):

L_{tot} :	52	53	54	55	56	57
NOS :	1	1	3	5	10	15



Edge states are described by:
$1\frac{1}{2}$ chiral phonon modes $c = 1\frac{1}{2}$
$=1$ chiral phonon mode
$+ 1$ chiral Majorana fermion
$=3$ chiral Majorana fermions **The Pfaffien state is non-Abelian**

Topo. order & theory of long range entanglement

The microscopic mechanism of superconductivity: electron pairing

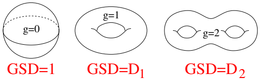
- The microscopic mechanism of topological order:
  Topological order = pattern of long range entanglement

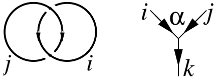
  Wen, PRB **40** 7387 (89); IJMPB 4, 239 (90). Chen Gu Wen arXiv:1004.3835

Symmetry breaking orders are described by group theory. What theory describes topological orders (long range entanglement)?

- **Ground states**: Robust degenerate ground states form vector bundles on moduli spaces of gapped Hamiltonians → **moduli bundle theory** for topological orders.

  Wen, IJMPB 4, 239 (90); Wen Niu PRB 41, 9377 (90)



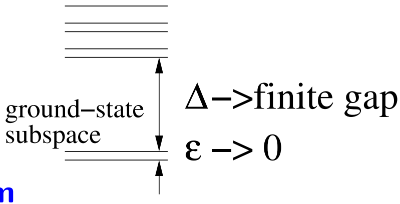


- **Excitations**: The anyons are described by their **fusion and braiding** → **modular tensor category theory** for topological orders

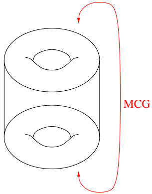
  Moore Seiberg CMP **123** 177 (89). Witten, CMP **121** 352 (89)

Moduli bundle theory of topological order

The important data is the **connections** of ground-state vector bundle on moduli space.



- Non-Abelian Berry's phase along **contractable loops** in moduli space $\to$ a diagonal $U(1)$ factor acting on the degenerate ground states $\to$ **gravitational Chern-Simons term** $\to$ **chiral central charge** c of edge state



- Non-Abelian Berry's phase along **non-contractable loops** in moduli space $\to$ S, T unitary matrices acting on the degenerate ground states $\to$ **projective representation of mapping-class-group** (which is $SL(2,\mathbb{Z})$ for torus, generated by $s : (x, y) \to (-y, x), \quad t : (x, y) \to (x + y, y)$)

Wen, PRB **40** 7387 (89); IJMPB 4, 239 (90).

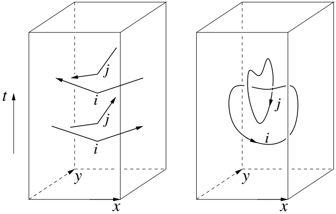
Modular tensor category theory for anyons and 2+1D topological orders



- Excitation in 2+1D topological order → **Braided fusion category (modular tensor category)** → A theory for 2+1D topological orders for bosons.
  rational CFT → TQFT → MTC
  Moore-Seiberg CMP **123** 177 (89); Witten, CMP **121** 352 (89)

- In higher dimensions, topological excitations can be **point-like**, **string-like**, *etc* , which can fuse and braid →
- Topological excitations are described by **non-degenerate braided fusion higher categories** → theory of topological order

- The ground state degeneracy *GSD* on torus and fractional statistics $\theta = \pi \frac{p}{q}$ of topological excitations are closely related $U_x U_y U_x^\dagger U_y^\dagger = e^{2\pi \frac{p}{q}}$: *GSD* is a multiple of q.



Wen Niu PRB **41** 9377 (90).

Classify 2+1D bosonic topological orders (TOs)

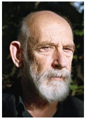
Using moduli bundle theory (*ie* $SL(2,\mathbb{Z})$ representations), plus input from modular tensor category, we can classify 2+1D bosonic topological orders (up to invertible $E(8)$ states):

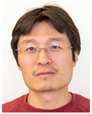
# of anyon types (rank)	1	2	3	4	5	6	7	8	9	10	11
# of 2+1D TOs	1	4	12	18	10	50	28	64	81	76	44
# of Abelian TOs	1	2	2	9	2	4	2	20	4	4	2
# of non-Abelian TOs	0	2	10	9	8	46	26	44	77	72	42
# of prime TOs	1	4	12	8	10	10	28	20	20	40	44

Rowell Stong Wang, arXiv:0712.1377: up to rank 4
Bruillard Ng Rowell Wang, arXiv:1507.05139: up to rank 5
Ng Rowell Wang Wen, arXiv:2203.14829: up to rank 6
Ng Rowell Wen, to appear: up to rank 11

- **This classifies all 2+1D gapped phases for bosonic systems without symmetry, with 11 topological excitations or less.**

Topological holographic principle






String holographic principle: Susskind hep-th/9409089
boundary CFT = bulk AdS gravity Maldacena hep-th/9711200

- **Holographic principle of topological order**:
  Boundary determines bulk, but bulk does not determine boundary Kong Wen arXiv:1405.5858; Kong Wen Zheng arXiv:1502.01690
  The excitations in a topological order are described by a braided fusion category $\mathcal{M}$. The excitations on a gapped boundary of a topological order are described by a fusion category $\mathcal{F}$
  $\mathcal{F}$ **determines** $\mathcal{M}$: $\mathcal{Z}(\mathcal{F}) = \mathcal{M}$ ($\mathcal{Z}$ is generalized Drinfeld-center)
  - String-operators that create pairs of boundary excitations form an algebra which is characterized by a braided fusion category $\mathcal{M}$.
  Chatterjee Wen arXiv:2205.06244
- **A generalization of anomaly in-flow**: Callan Harvey, NPB **250** 427 (1985)
  The theory described by fusion category $\mathcal{F}$ has a (non-invertible) gravitational anomaly (*ie* no UV completion) Kong Wen arXiv:1405.5858
  (non-invertible) grav anomaly = bulk topological order $\mathcal{M}$

Classification of 3+1D bosonic topological orders (*ie* classification of 4D fully extended TQFTs)

An application of topological holographic principle

- 3+1D bosonic topological orders with only bosonic point-like excitations are classified by 3+1D Dijkgraaf-Witten theory of finite groups. Lan Kong Wen arXiv:1704.04221; Johnson-Freyd arXiv:2003.06663

- 3+1D fully extended TQFT's with only bosonic point-like excitations are classified by Dijkgraaf-Witten theories of finite groups.

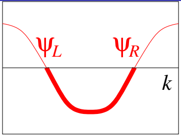
- A duality relation: 3+1D twisted higher gauge theories of finite higher group with only bosonic point-like excitations are equivalent to twisted 1-gauge theories of finite group.

- 3+1D bosonic topological orders with both bosonic and fermionic point-like excitations are also classified.

Lan Wen arXiv:1801.08530; Johnson-Freyd arXiv:2003.06663

Next step: a general theory for 'finite' gapless state

A gapless state has emergent (and exact) symmetry:



- Group-like symmetries Heisenberg, Wigner, 1926 $U(2) \to$
- Anomalous symmetries 't Hooft, 1980 $U_R(2) \times U_L(2)$
- Higher-form symmetries Nussinov Ortiz 09; Gaiotto Kapustin Seiberg Willett 14
- Higher-group symmetries Kapustin Thorngren 2013
- Algebraic higher symmetry Thorngren Wang 19; Kong Lan Wen Zhang Zheng 20

  algebraic (higher) symmetry = non-invertible (higher) symmetry = fusion (higher) category symmetry = ...

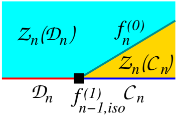
  Petkova Zuber 2000; Coquereaux Schieber 2001; ... for 1+1D CFT
- (Non-invertible) gravitational anomalies Kong Wen 2014; Ji Wen 2019

- Conjecture: **The maximal emergent (generalized) symmetry largely determine the gapless states.**

  A classification of maximal emergent (generalized) symmetries → A classification of "finite" gapless states. Chatterjee Ji Wen arXiv:2212.14432

  What is the general theory for all those generalized symmetries, which are beyond group and higher group?

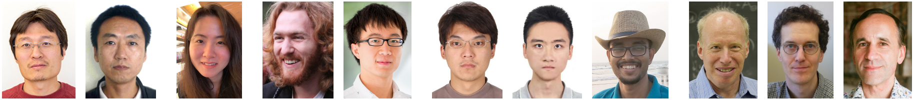
Symmetry/Topological-Order correspondence



A symmetry corresponds to:

- an **isomorphic decomposition** $\mathcal{D}_n \cong \mathcal{C}_n \boxtimes_{\mathcal{Z}_n(\mathcal{C}_n)} f_n^{(0)}$
  Kong Wen Zheng arXiv:1502.01690; Freed Moore Teleman arXiv: 2209.07471
- a **non-invertible gravitational anomaly** Ji Wen arXiv:1905.13279
- a **symmetry + dual symmetry + braiding** Ji Wen arXiv:1912.13492
  Conservation/fusion-ring of **symmetry charges** = symmetry
  Conservation/fusion-ring of **symmetry defects** = dual-symmetry
- a **gappable-boundary topological order** in one higher dimension
  Ji Wen arXiv:1912.13492; Kong Lan Wen Zhang Zheng arXiv:2005.14178
- a **Braided fusion higher category in trivial Witt class**
  Thorngren Wang arXiv:1912.02817; Kong Lan Wen Zhang Zheng arXiv:2005.14178.

  → a unified frame work to classify SSB, TO, SPT, SET phases.
- a **topological skeleton** in QFT Kong Zheng arXiv:2011.02859
- an **algebra of patch commutant operators.**
  Kong Zheng arXiv:2201.05726; Chatterjee Wen arXiv:2205.06244



Symmetry $\sim$ non-invertible gravitational anomaly

- A symmetry is generated by an unitary operators U that commute with the Hamiltonian: $UH = HU$.
- We describe a symmetric system (with lattice UV completion) restricted in the symmetric sub-Hilbert space

$$U\mathcal{V}_{\text{symmetric}} = \mathcal{V}_{\text{symmetric}}.$$

Both system and the probing instruments respect the symmetry

- The symmetry transformation U acts trivially within $\mathcal{V}_{\text{symmetric}}$.

How to know there is a symmetry? How to identify the symmetry?

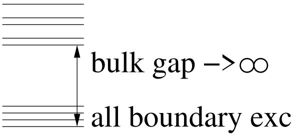
- The total Hilbert space $\mathcal{V}_{\text{tot}}$ has a tensor product decomposition $\mathcal{V}_{\text{tot}} = \otimes_i \mathcal{V}_i$, where i labels sites, due to the lattice UV completion.
- The symmetric sub-Hilbert space $\mathcal{V}_{\text{symmetric}}$ does not have a tensor product decomposition $\mathcal{V}_{\text{symmetric}} \neq \otimes_i \mathcal{V}_i$, indicating the presence of a symmetry.
- Lack of tensor product decomposition $\to$ gravitational anomaly. $\to$ **symmetry $\cong$ non-invertible gravitational anomaly**

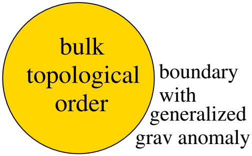
Symmetry ≅ topological order in one higher dim

- **Gravitational anomaly = topo. order in one higher dim**

- The total boundary Hilbert space of a topologically ordered state has no tensor product decomposition. Yang etal arXiv:1309.4596 Lack of tensor product decomposition is described by boundary of topological order

Kong Wen arXiv:1405.5858





Systems with a (generalized) symmetry (restricted within $\mathcal{V}_{\text{symmetric}}$) can be fully and exactly simulated by boundaries of a topological order, called **symmetry-TO** (with lattice UV completion) or **symmetry TFT**.

Ji Wen arXiv:1912.13492; Kong Lan Wen Zhang Zheng arXiv:2005.14178

Apruzzi Bonetti Etxebarria Hosseini Schafer-Nameki arXiv:2112.02092

- Symmetry-TO or symmetry TFT was originally called **categorical symmetry** in Ji Wen arXiv:1912.13492; Kong etal arXiv:2005.14178

→ **Symm/TO correspondence**

Classify 1+1D symmetries (up to holo-equivalence)

Not every topological order describes a generalized symmetry.

- Only topological orders with gappable boundary (*ie* in trivial Witt class) correspond to (generalized) symmetries.

Kong Lan Wen Zhang Zheng arXiv:2005.14178; Freed Moore Teleman arXiv:2209.07471

We refer to gappable-boundary topological order (TO) in one higher dimension as **symmetry-TO** (with lattice UV completion).

Finite symmetries (up to holo-equivalence) are one-to-one classified by symmetry-TOs in one higher dimension

- We can use 2+1D symmetry-TOs (instead of groups) to classify 1+1D finite (generalized) symmetries (up to holo-equivalence):

# of symm charges/defects (rank)	1	2	3	4	5	6	7	8	9	10	11
# of 2+1D TOs	1	4	12	18	10	50	28	64	81	76	44
# of symm classes (symm-TOs)	1	0	0	3	0	0	0	6	6	≤ 3	0
# of (anomalous) group-symmetries	$1_{\mathbb{Z}_1}$	0	0	$2_{\mathbb{Z}_2^\omega}$	0	0	0	$6_{S_3^\omega}$	$3_{\mathbb{Z}_3^\omega}$	0	0

- At rank-4: $\mathbb{Z}_2$ symm, anomalous $\mathbb{Z}_2$ symm, double-Fibonacci symm

Local fusion category & isomorphic decomposition

An anomaly-free ordineray symmetry is decribed by a group

- An anomaly-free generalized (*ie* non-invertible higher) symmetry (*ie* algebraic higher symmetry) in $n+1$D is decribed by

- a **local fusion n-category** $\mathcal{R}_{\text{charge}}$ that describes symmetry charges (excitations over trivial symmetric ground state), or by

- a **local fusion n-category** $\widetilde{\mathcal{R}}_{\text{defect}}$ that describes symmetry defects.

Thorngren Wang arXiv:1912.02817 (1+1D); Kong Lan Wen Zhang Zheng arXiv:2005.14178

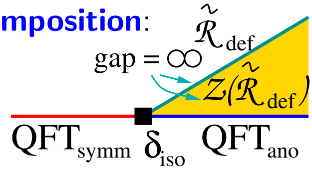
- **generalized symmetry = isomorphic decomposition**:

δ_{iso} : $QFT_{\text{symm}} \cong QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}}$

Kong Wen Zheng arXiv:1502.01690

Kong Lan Wen Zhang Zheng arXiv:2005.14178

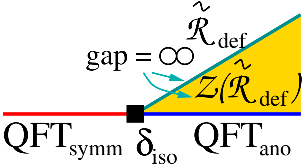
δ_{iso} : $Z(QFT_{\text{symm}}) = Z(QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\widetilde{\mathcal{R}}_{\text{def}})} \widetilde{\mathcal{R}}_{\text{def}})$



- **A similar but different theory**: A generalized (potentially anomalous) symmetry $= (\rho, \sigma = \mathcal{Z}(\rho)) =$ fusion n-categroy ρ (**no local condition**). Freed Moore Teleman arXiv: 2209.07471

Classify gapped/gapless phases of symm systems

via Symm/TO correspondence:



δ_{iso} : $QFT_{\text{symm}} \cong QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})} \tilde{\mathcal{R}}_{\text{def}}$

Kong Wen Zheng arXiv:1502.01690

Kong Lan Wen Zhang Zheng arXiv:2005.14178

δ_{iso} : $Z(QFT_{\text{symm}}) = Z(QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})} \tilde{\mathcal{R}}_{\text{def}})$

- Gapped liquid phases are gapped boundaries of $\mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})$(symm-TO)

- Includes spontaneous symmetry breaking orders, symmetry protected topological (SPT) orders, symmetry enriched topological (SET) orders for systems with algebraic higher symmetry $\tilde{\mathcal{R}}_{\text{def}}$

- Gapless liquid phases are gapless boundaries of $\mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})$ (symm-TO)
- SPT phases protected by algebraic higher symmetry $\tilde{\mathcal{R}}_{\text{def}}$ are classified by the automorphisms α of the corresponding symmetry-TO $\mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})$, that leave $\tilde{\mathcal{R}}_{\text{def}}$ invariant.
- Anomalous algebraic higher symmetries are classified by $(\tilde{\mathcal{R}}_{\text{def}}, \tilde{\alpha})$, where $\tilde{\alpha} \in \text{Auto}(\mathcal{Z}(\Sigma\tilde{\mathcal{R}}_{\text{def}}))$ that leave $\Sigma\tilde{\mathcal{R}}_{\text{def}}$ invariant.

A general theory of duality (holo-equivalence)

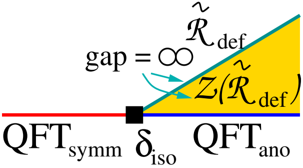
via Symm/TO correspondence and isomorphic decomposition:

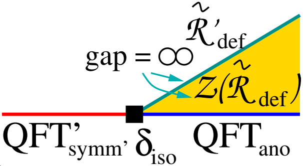
$\delta_{\text{iso}}: \ QFT_{\text{symm}} \cong QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})} \tilde{\mathcal{R}}_{\text{def}}$

Kong Wen Zheng arXiv:1502.01690

Kong Lan Wen Zhang Zheng arXiv:2005.14178

$\delta_{\text{iso}}: \ Z(QFT_{\text{symm}}) = Z(QFT_{\text{ano}} \boxtimes_{\mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}})} \tilde{\mathcal{R}}_{\text{def}})$





- Choose a different gapped boundary $\tilde{\mathcal{R}}'_{\text{def}}$, without changing the bulk topological order $\mathcal{Z}(\tilde{\mathcal{R}}_{\text{def}}) = \mathcal{Z}(\tilde{\mathcal{R}}'_{\text{def}})$ and without changing the boundary QFT_{anom} $\to$ the two quantum field theories, QFT_{symm} and $\text{QFT'}_{\text{symm'}}$, are holo-equivalent, or are related by **duality** or **gaugimg** transformation. Bhardwaj Tachikawa arXiv:1704.02330

- QFT_{symm} and $\text{QFT'}_{\text{symm'}}$ may have different generalized symmetries.

- Two generalized symmetries $\tilde{\mathcal{R}}$ and $\tilde{\mathcal{R}}'$ are holo-equivalent, if they have the same bulk (*ie* the same symmetry-TO) $\mathcal{Z}(\tilde{\mathcal{R}}) = \mathcal{Z}(\tilde{\mathcal{R}}')$.

- 1+1D $\mathbb{Z}_2 \times \mathbb{Z}_2$ symmetry with mixed anomaly $\cong \mathbb{Z}_4$ symmetry

Gapped/gapless phases of symmetric systems are ‘classified’ by condensible algebras of symmetry-TO

- For **1+1D systems** with (generalized) symmetry, their gapped states and gapless states can be “classfied” by **condensible algebras** $\mathcal{A} = \mathbf{1} \oplus a \oplus b ...$ (*ie* the sets of anyons that can condense together) in the corresponding symmetry-TO (in one higher dimension):

- **The maximal (Langrangian) condensible algebras of the 2+1D symmetry-TO classify (1-to-1) gapped phases.**
- **The non-maximal (non-Langrangian) condensible algebras of the 2+1D symmetry-TO label (1-to-many) gapless phases (1+1D CFTs).**

This is because the gappled/gapless boundaries of 2+1D topological orders $\mathcal{M}$ are “classified” by the condensible algebras $\mathcal{A}$ of $\mathcal{M}$.

Classify 1+1D gapped phases for systems w/ $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ symm via Lagrangian condensable algebra

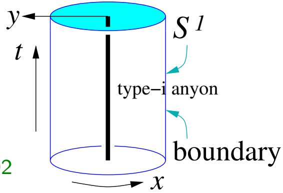
- The symmetry-TO for 1+1D $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ symmetry is 2+1D $\mathbb{Z}_2^a \times \mathbb{Z}_2^b$ gauge theory $\mathcal{G}au_{\mathbb{Z}_2^a \times \mathbb{Z}_2^b}$, with excitations **generated by** e_a, e_b, m_a, m_b.
- Six Lagrangian condensible algebras: Chatterjee Wen arXiv:2205.06244

$$
\begin{aligned}
\mathbf{1} \oplus m_a \oplus m_b \oplus m_a m_b &\to \mathbb{Z}_2^a\text{-symmetric-}\mathbb{Z}_2^b\text{-symmetric} \\
\mathbf{1} \oplus m_a \oplus e_b \oplus m_a e_b &\to \mathbb{Z}_2^a\text{-symmetric-}\mathbb{Z}_2^b\text{-broken} \\
\mathbf{1} \oplus e_a \oplus m_b \oplus e_a m_b &\to \mathbb{Z}_2^a\text{-broken-}\mathbb{Z}_2^b\text{-symmetric} \\
\mathbf{1} \oplus e_a \oplus e_b \oplus e_a e_b &\to \mathbb{Z}_2^a\text{-broken-}\mathbb{Z}_2^b\text{-broken} \\
\mathbf{1} \oplus e_a e_b \oplus m_a m_b \oplus e_a m_a e_b m_b &\to \text{diagonal-}\mathbb{Z}_2\text{-symmetric} \\
\mathbf{1} \oplus e_a m_b \oplus m_a e_b \oplus e_a m_a e_b m_b &\to \mathbb{Z}_2^a \times \mathbb{Z}_2^b \text{ SPT phase}
\end{aligned}
$$

Q: How symmetry-TO determines gapless states? A: Via modular covariant partition function

A symmetry is described by its symmetry-TO. Its gapless states are simulated by the boundaries of the symmetry-TO.

- Boundary of 2+1D symmetry-TO has a **vector-valued partition function**, whose component $Z_i(\tau, \bar\tau)$ is labeled by the anyon types i of the 2+1D bulk topological order.
  Chen *etal* arXiv:1903.12334; Ji Wen arXiv:1905.13279, 1912.13492
  Kong Zheng arXiv:1905.04924, arXiv:1912.01760



- $Z_i(\tau, \bar\tau)$ is not modular invariant but **modular covariant**:

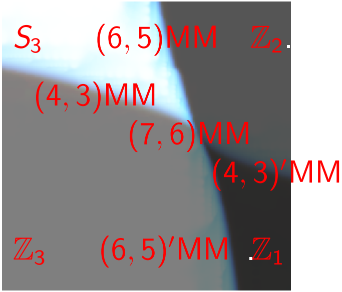
$$T^{\mathcal{M}}: \; Z_i(\tau+1) = T^{\mathcal{M}}_{ij} Z_j(\tau), \qquad S^{\mathcal{M}}: \; Z_i(-1/\tau) = S^{\mathcal{M}}_{ij} Z_j(\tau).$$

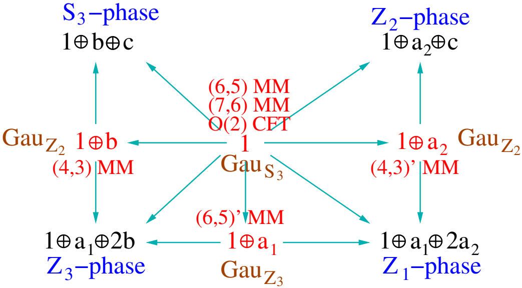
  where $S^{\mathcal{M}}$, $T^{\mathcal{M}}$-matrix characterize the 2+1D bulk topological order $\mathcal{M}$ (*ie* the symmetry-TO).
  Ji Wen arXiv:1905.13279, 1912.13492; Lin Shao arXiv:2101.08343

- **CFT (gapless liquid phase) is a number theoretical problem.**

The symmetry-TO for 1+1D S_3 symmetry is 2+1D S_3-gauge theory $\mathcal{G}au_{S_3}$ → gapped/gapless states

d,s	$1,0$	$1,0$	$2,0$	$2,0$	$2,\frac{1}{3}$	$2,-\frac{1}{3}$	$3,0$	$3,\frac{1}{2}$
$\otimes$	$\mathbf{1}$	a_1	a_2	b	b_1	b_2	c	c_1
$\mathbf{1}$	$\mathbf{1}$	a_1	a_2	b	b_1	b_2	c	c_1
a_1	a_1	$\mathbf{1}$	a_2	b	b_1	b_2	c_1	c
a_2	a_2	a_2	$\mathbf{1}\oplus a_1\oplus a_2$	$b_1\oplus b_2$	$b\oplus b_2$	$b\oplus b_1$	$c\oplus c_1$	$c\oplus c_1$
b	b	b	$b_1\oplus b_2$	$\mathbf{1}\oplus a_1\oplus b$	$b_2\oplus a_2$	$b_1\oplus a_2$	$c\oplus c_1$	$c\oplus c_1$
b_1	b_1	b_1	$b\oplus b_2$	$b_2\oplus a_2$	$\mathbf{1}\oplus a_1\oplus b_1$	$b\oplus a_2$	$c\oplus c_1$	$c\oplus c_1$
b_2	b_2	b_2	$b\oplus b_1$	$b_1\oplus a_2$	$b\oplus a_2$	$\mathbf{1}\oplus a_1\oplus b_2$	$c\oplus c_1$	$c\oplus c_1$
c	c	c_1	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$\mathbf{1}\oplus a_2\oplus b\oplus b_1\oplus b_2$	$a_1\oplus a_2\oplus b\oplus b_1\oplus b_2$
c_1	c_1	c	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$a_1\oplus a_2\oplus b\oplus b_1\oplus b_2$	$\mathbf{1}\oplus a_2\oplus b\oplus b_1\oplus b_2$





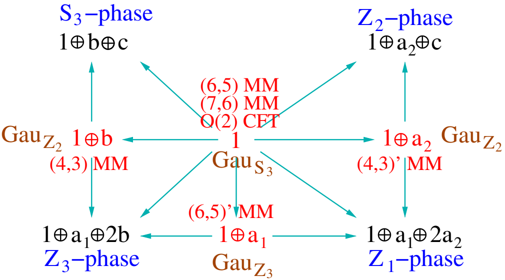
Chatterjee Wen arXiv:2205.06244

The S_3 Symmetry-TO $\mathcal{G}au_{S_3}$ has an automorphism

d,s	$1,0$	$1,0$	$2,0$	$2,0$	$2,\frac{1}{3}$	$2,-\frac{1}{3}$	$3,0$	$3,\frac{1}{2}$
$\otimes$	$\mathbf{1}$	a_1	a_2	b	b_1	b_2	c	c_1
$\mathbf{1}$	$\mathbf{1}$	a_1	a_2	b	b_1	b_2	c	c_1
a_1	a_1	$\mathbf{1}$	a_2	b	b_1	b_2	c_1	c
a_2	a_2	a_2	$\mathbf{1}\oplus a_1\oplus a_2$	$b_1\oplus b_2$	$b\oplus b_2$	$b\oplus b_1$	$c\oplus c_1$	$c\oplus c_1$
b	b	b	$b_1\oplus b_2$	$\mathbf{1}\oplus a_1\oplus b$	$b_2\oplus a_2$	$b_1\oplus a_2$	$c\oplus c_1$	$c\oplus c_1$
b_1	b_1	b_1	$b\oplus b_2$	$b_2\oplus a_2$	$\mathbf{1}\oplus a_1\oplus b_1$	$b\oplus a_2$	$c\oplus c_1$	$c\oplus c_1$
b_2	b_2	b_2	$b\oplus b_1$	$b_1\oplus a_2$	$b\oplus a_2$	$\mathbf{1}\oplus a_1\oplus b_2$	$c\oplus c_1$	$c\oplus c_1$
c	c	c_1	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$\mathbf{1}\oplus a_2\oplus b\oplus b_1\oplus b_2$	$a_1\oplus a_2\oplus b\oplus b_1\oplus b_2$
c_1	c_1	c	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$c\oplus c_1$	$a_1\oplus a_2\oplus b\oplus b_1\oplus b_2$	$\mathbf{1}\oplus a_2\oplus b\oplus b_1\oplus b_2$

Chatterjee Wen arXiv:2205.06244

The $\mathcal{G}au_{S_3}$ symmetry-TO is invariant under the exchange $a_2 \leftrightarrow b$, which corresponds to a horizontal reflections of the S_3 phase diagram:



(S_3-phase, $\mathbb{Z}_3$-phase) $\leftrightarrow$ ($\mathbb{Z}_2$-phase, $\mathbb{Z}_1$-phase)

Automorphism in Symm-TO→equivalent transition

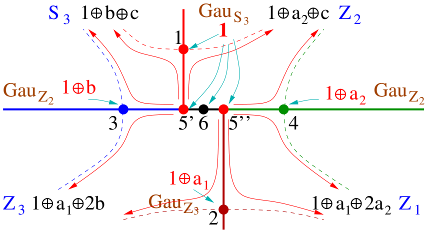
Chatterjee Wen arXiv:2205.06244

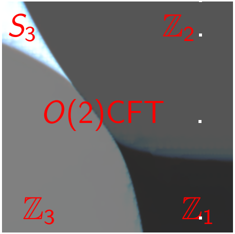
- The phase transitions $S_3 \leftrightarrow \mathbb{Z}_1$ and $\mathbb{Z}_3 \leftrightarrow \mathbb{Z}_2$ are equivalent.
- The phase transitions $S_3 \leftrightarrow \mathbb{Z}_3$ and $\mathbb{Z}_2 \leftrightarrow \mathbb{Z}_1$ are equivalent.
- The following two paires of multi-critical points are equivalent

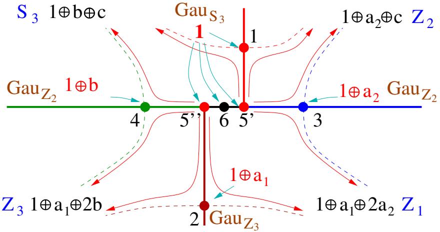
$(S_3, \mathbb{Z}_3, \mathbb{Z}_2) = (S_3, \mathbb{Z}_2, \mathbb{Z}_1)$

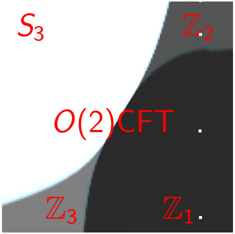
$(\mathbb{Z}_3, \mathbb{Z}_2, \mathbb{Z}_1) = (S_3, \mathbb{Z}_3, \mathbb{Z}_1)$

They are **1**-condensed CFT with 2 relavent operators and has $(c, \bar{c}) = (1, 1)$



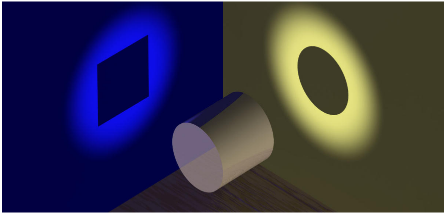


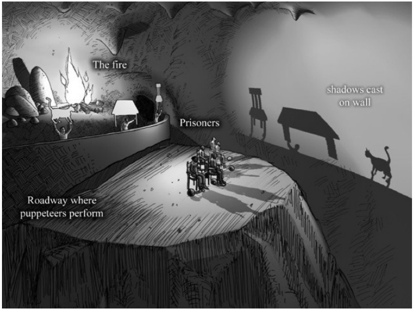




The essence of a symmetry

- Emergent symmetries can go beyond groups, higher groups, and/or anomalies. But their can always be described by **a gappable-boundary topological order in one higher dimension** (with lattice UV completion) = **symmetry-TO**





- The same topological order (in one higher dimensions) can have different shadows $\rightarrow$ **holo-equivalent symmetries**.

Category $\leftrightarrow$ **Generalized symmetry** $\leftrightarrow$ **Geometry/CFT**