# Discussion Session: Strings, QFT, and Math

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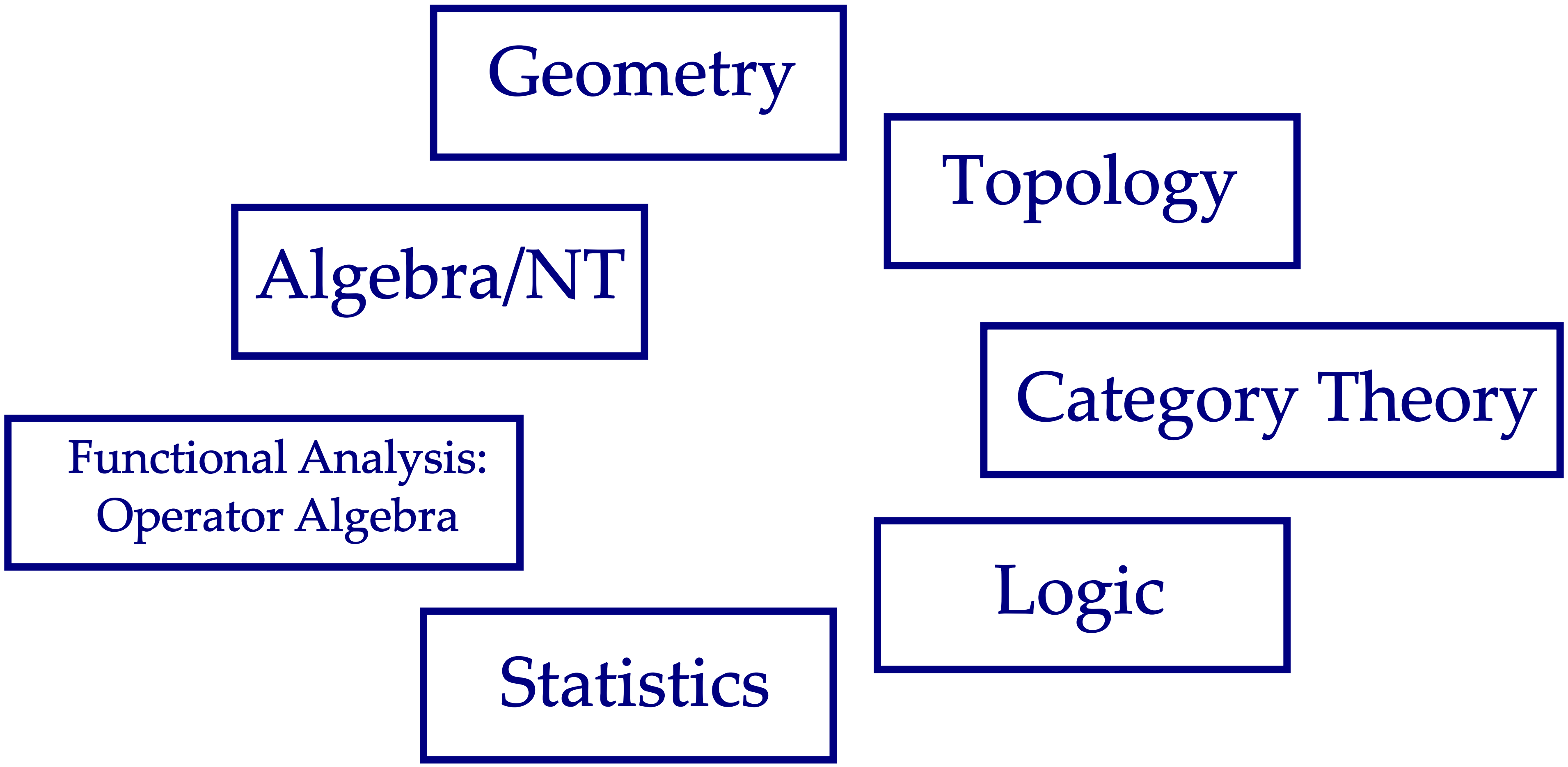
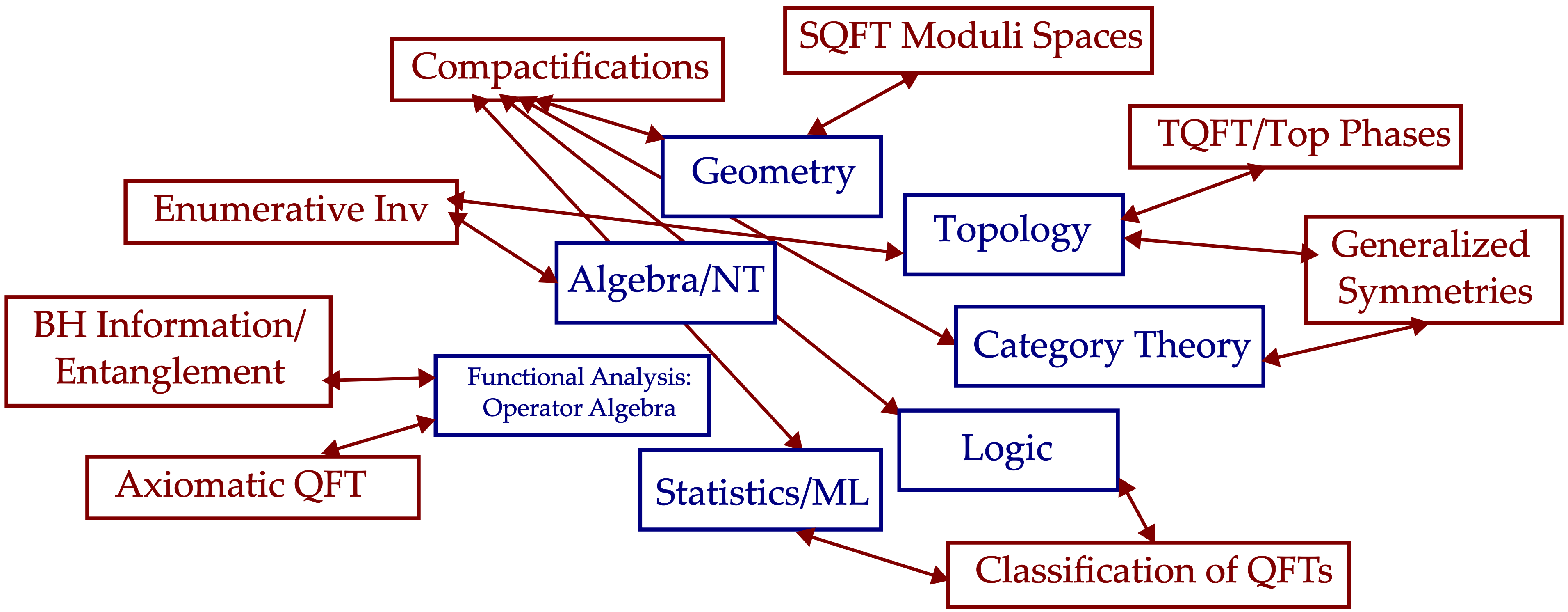
The Fields of Mathematics



The Fields of Mathematics... and some of their connections to String Theory/QFT:




Many other connections (probably complete graph).

See [Frenkel]'s talk, and [Snowmass/A Panorama Of Physical Mathematics c. 2022: Bah, Freed, Moore, Nekrasov, Razamat, SSN]

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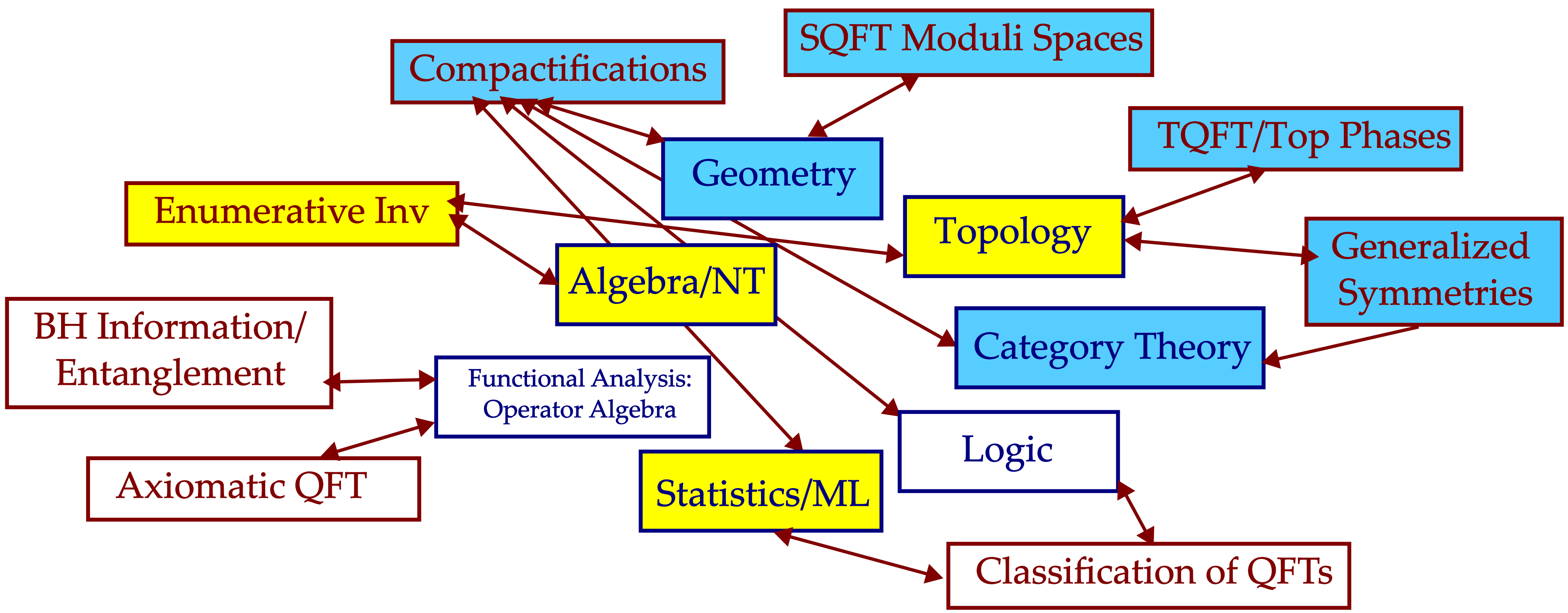


Two connections – biased selection – that have seen a lot of progress in the past few years:

1. Geometry and QFTs: Geometric Engineering
2. Category Theory and Generalized Symmetry

## 1. Geometry and String Theory/QFT

String Compactifcations: Geometry encoding properties of low energy effective theories:

# QFT (non-compact)

# QG (compact) (see swampland talks).

**QFT - Geometry Connection:**

1. 8 supercharges:
   geometric classification of superconformal theories (SCFTs):
   Geometries are usually fairly well understood (singular CY):

   **Algebraic Geometry ⇒ QFT**

   - 6d: F-theory on elliptic Calabi-Yau threefold (CY3) classification (modulo frozen phases)
   - 5d: M-theory on canonical CY3 singularities classification (in theory)
   - 4d: IIB on canonical CY3 singularities constructions; classification?
   - 3d: reductions from 5d and 4d, i.e. M on CY3$\times T^2$ or IIB on CY3 $\times S^1$. Geometric realization of 3d mirror symmetry open problem in general

2. 4 supercharges:
   Geometries are far less well understood. In fact in past years, physics motivation for new geometries were made:

   **QFT $\Rightarrow$ Geometry**

   - 4d QFTs: M-theory on $G_2$ holonomy some recent geometric progress of explicit constructions, using insights from physics (M/IIA reduction leads to new collapsed limits of $G_2$ spaces to CY3)
     Compact $G_2$: codimension 7 singularity still wide open problem.
   - 4d SCFTs: M-theory on $G_2$ holonomy new conjectured constructions of $G_2$ motivated from reductions of QFTs:
     5d SCFT with 4d $\mathcal{N} = 1$ domain walls
     6d SCFT reductions with fluxes and punctures on $M_2$
     Proof that these constructions admit torsion-free $G_2$ structures.
     First principle geometric criterion for conformal invariance?

Lots of progress, and dialog goes both ways: **Geometry $\Longleftrightarrow$ QFT/Strings**.

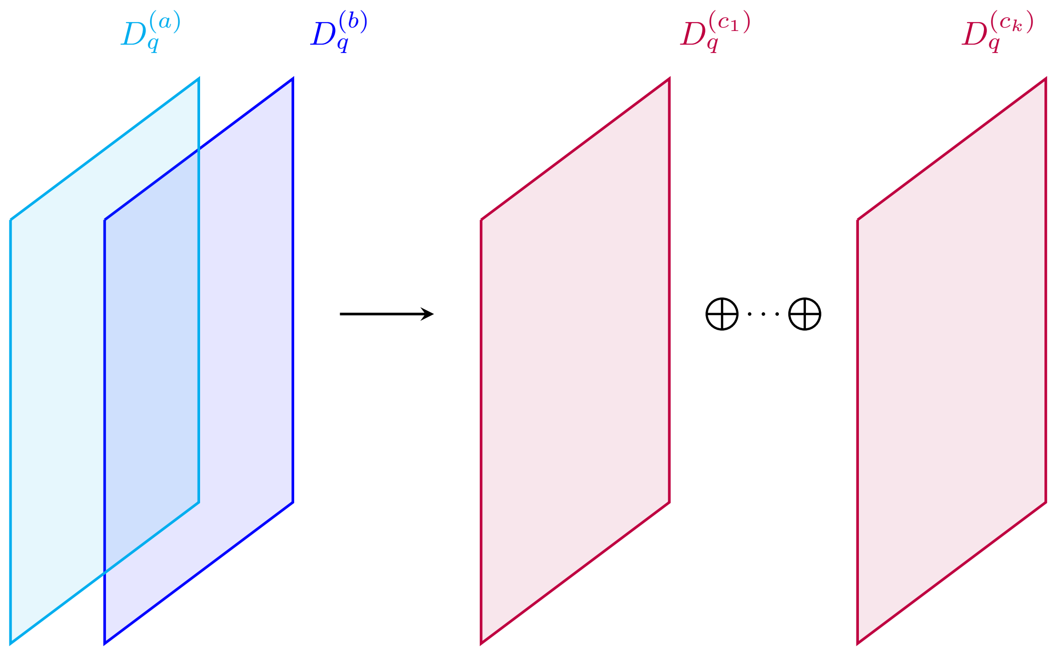
## 2. Category Theory and Generalized Symmetries

- Global symmetries correspond to topological sectors of QFTs [Gaiotto, Kapustin, Seiberg, Willett]
- Codim $p+1$ topological defects $D_{d-p-1}$ generate $p$-form symmetry $G^{(p)}$
- Composition/Fusion: Higher-form symmetry groups

$$D^{(g)}_{d-p-1} \otimes D^{(h)}_{d-p-1} = D^{(gh)}_{d-p-1}$$

More generically:
groups replaced by algebras ("non-invertible" symmetries), or more precisely higher-categories

$$D^{(a)}_{d-p-1} \otimes D^{(b)}_{d-p-1} = \bigoplus_c n^c_{ab} D^{(c)}_{d-p-1}$$



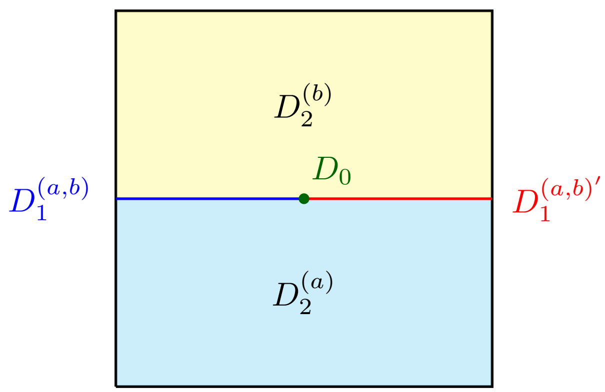
$D_q^{(a)}$
$D_q^{(b)}$
$D_q^{(c_1)}$
$\oplus \cdots \oplus$
$D_q^{(c_k)}$

Such non-invertible symmetries are ubiquitous in very standard QFTs and generally give rise to:

Higher-categorical Symmetries.

In a $d$-dimensional QFT, there can be topological operators $D_q$ of dimensions $q = 0, \cdots, d-1$. Each layer has a fusion, which in general is non-invertible: fusion $(d-1)$-category.

E.g. 3d QFT has topological surfaces $D_2$, lines $D_1$ and point operators $D_0$ forming a 2-category:



# Fusion Higher-Categorical Symmetries

Examples:

- $n\text{Vec}_G^\omega$ for $G$ finite group corresponds to group-like $n$-dimensional topological defects, or generally a higher-group: $\omega$ generalizes anomalies.
- $n\text{Rep}(G)$ generalizes the category of representations: non-invertible ($n > 1$, and $n = 1$ require $G$ representations)
- Self-duality defects

Classification of possible symmetries translates into classification of fusion higher-categories, including higher-structures, like associators

Lots of very recent math results on fusion higher-categories [Douglas–Reutter, Kong, Johnson-Freyd, Gaiotto, Décoppet, Yu,...] on higher fusion categories, and in physics [Bhardwaj, Bottini, SSN, Tiwari, Bartsch, Bullimore, Ferrari, Wu, $\cdots$ ]

# Challenge 1: Generalized Charges

What generalizes representations of a symmetry group?

**How do categorical symmetries act on physical (not necessarily topological) operators?**

Denote $q$-dim extended operators charged under a symmetry as $q$-charges.

- Even for invertible $G^{(p)}$ symmetries:

  **$q$-charges of a $p$-form symmetry $G^{(p)}$ form $(q+1)$ -representations of $G^{(p)}$**

- For non-invertible symmetries: [Bhardwaj, SSN]

  **$q$-charges are the topological defects of the $(d+1)$-dim Symmetry Topological Field theory (SymTFT)**

  Mathematically the Drinfeld center of the symmetry category: very well developed for 2-categories.

Challenges:

Develop ”representation theory” of categorical symmetries, i.e. determine $q$-charges, and their realization in QFTs; for $n \geq 3$-categories: computing Drinfeld centers.

## Challenges 2: Physical Implications

What are new insights that can be gained from these symmetries? Anomalies?
E.g. some simple applications so far classification of gapped phases preserving certain non-invertible symmetries.
Inspiration from cond-mat – see [Wen]'s talk.

## Challenges 3: Are there any interesting implications for Category Theory?

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