Revisiting logarithmic corrections to supersymmetric black hole entropy

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A.H. Anupam, P.V. Athira, Chandramouli Chowdhury, A.S., arXiv:2306.07322
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Bekenstein-Hawking formula for black hole entropy is universal

$$S_0 = \frac{A}{4} \qquad \text{in } \hbar = c = G_N = k_B = 1 \text{ unit}$$

A: area of the event horizon

In Einstein-Maxwell theory or (extended) supergravity in D dimensions, the black hole can carry

– U(1) charges Q_k

– angular momentum J_i in Cartan subalgebra of SO(D-1)

– mass M

Then

$$S_0 = f_0(Q, M, J)$$

Q, J have multiple components in general

Scaling property of the entropy in D dimensions

$$f_0(\lambda^{D-3}Q, \lambda^{D-3}M, \lambda^{D-2}J) = \lambda^{D-2}\, f_0(Q, M, J)$$

To take macroscopic limit, we take

$$M \sim \lambda^{D-3}, \qquad Q \sim \lambda^{D-3}, \qquad J \sim \lambda^{D-2}$$

and take λ large

Then

$$S_0 \sim \lambda^{D-2}$$

In this limit the fields associated with the black hole also has simple dependence on λ, e.g.

$$g_{\mu\nu} \sim \lambda^2$$

Note: In D=4 we can also have magnetic charges scaling as λ, but they are topological and will not play any role in this analysis

The Bekenstein-Hawking formula is expected to receive corrections due to stringy effects and quantum effects

General structure:

$$\mathbf{S} = \lambda^{\mathbf{D}-2}\,\mathbf{f_0} + \textbf{power suppressed terms} + \mathbf{C}\,(\ln\lambda) + \cdots$$

Focus of attention in today's lecture will be the terms $\propto \ln\lambda$

General procedure for computing corrections to the black hole entropy (Gibbons-Hawking)

1. Perform a path integral over all fields subject to the same boundary condition that the black hole satisfies

– gives partition function containing ln λ corrections

Fursaev, Solodukhin, $\cdots$, Review: arXiv:1104.3712 by Solodukhin

2. Construct the entropy from the partition function using the usual rules of statistical mechanics

e.g. for asymptotically flat black holes, the gravitational path integral gives grand canonical partition function

– need to take appropriate Laplace transform to get the microcanonical entropy

Can also generate logarithmic corrections to the entropy

A.S.: arXiv:1205.0971

Example: Result for Kerr black hole in pure gravity in D=4

$$\left(\frac{212}{45} - 1\right) \ln \lambda$$

Any quantum theory of gravity that can count black hole microstates should reproduce this result.

At present such a counting is not possible in string theory.

This can be remedied using supersymmetric black holes.

Supersymmetric (extremal) black holes have zero temperature

$\Rightarrow$ instead of having a single large length scale, we have two different large scales

M, Q $\sim \lambda^{D-3}$ and $\beta \equiv \frac{\partial S}{\partial M} \to \infty$

– difficult to extract log correction

Remedy: Work in the near horizon geometry:

$AdS_2 \times$ (squashed) S^{D-2} Mann, Solodukhin hep-th/9604118; $\cdots$

$$ds^2 = v_1 \left(\frac{dr^2}{r^2 - 1} + (r^2 - 1) d\tau^2 \right) + v_2 \, ds^2_{D-2}$$

$v_1, v_2 \sim \lambda^2$

We can compute logarithmic correction to the partition function in this geometry following the same guidelines

Some differences:

1. The partition function computes the path integral at fixed mass, charge and angular momentum since these modes dominate as r $\to \infty$

$\Rightarrow$ the path integral directly computes the entropy in the microcanonical ensemble and no change of ensemble is needed.

2. We integrate over modes living in the near horizon geometry

– different set of eigenvalues and eigenfunctions than those in the full geometry

Final result in theories with N $\geq$ 2 supersymmetry in D=4:

$$S = S_0 + \frac{1}{6}\,(23 + n_H - n_V)\ \ln\lambda \quad \text{for N=2}$$

n_H, n_V: number of vector and hypermultiplets

$$S = S_0 \quad \text{for N=4}$$

$$S = S_0 - 8\ln\lambda \quad \text{for N=8}$$

Banerjee, Gupta, A.S. arXiv:1005.3044, Banerjee, Gupta, Mandal, A.S. arXiv:1106.0080

A.S. arXiv:1108.3842

The results are in perfect agreement with microscopic counting formula for N=4 CHL type compatifications and N=8 compactifications

Maldacena, Moore, Strominger hep-th/9903163; Dijkgraaf, Verlinde, Verlinde hep-th/9607026;

David, A.S. hep-th/0605210; David, Jatkar, A.S. hep-th/0609109

Similar agreement also holds in D=5 A.S. arXiv:1109.3706

No microscopic counting exists for black holes in N=2 theories

Recent developments

Iliesiu, Kologlu and Turiaci described a procedure for computing supersymmetric index using full black hole geometry

Iliesiu, Kologlu, Turiaci arXiv:2107.09062

Our goal will be to use this formalism to compute logarithmic correction to the black hole entropy

Supersymmetric index:

$$I = Tr_{Q,J',k=0}\left[e^{-\beta H}(-1)^F (2J_0)^n\right]$$

J_0 some particular Cartan generator, k: momentum

J' represents Cartan generators other than J_0

The trace is taken over states at fixed Q, J' and k=0

The trace gets contribution from only those states that break 2n (or less) J'-invariant supersymmetries

A generic non-BPS state will break all ($> 2n$) supersymmetries and will not contribute to this index for sufficiently small n

This index is expected to pick up the degeneracy of the supersymmetric states with fixed $Q, J', k = 0$

Bachas, Kiritsis hep-th/9611205; Gregori, Kiritsis, Kounnas, Obers, Petropoulos, Pioline hep-th/9708062

Dabholkar, Gomes, Murthy, A.S.

$$I = Tr_{Q,J',k=0}\left[e^{-\beta H}(-1)^F(2J_0)^n\right] \equiv e^{S_{BPS}-\beta\, M_{BPS}}$$

Examples:

1. In D=4 the rotation group is SU(2)

J_0 is the third generator of the rotation group, J' trivial

The corresponding microscopic index is counted in $N = 4, 8$ supersymmetric theories

2. In D=5 the rotation group is $SO(4) = SU(2)_L \times SU(2)_R$

We can take $J_0 = J_{3R}$, $J' = J_{3L}$

Index is a function of J_{3L} and electric charges

– counted in N=2, 4 supersymmetric compactifications (BMPV black holes)

Index from gravitational path integral

Euclidean continuation of a black hole leads to a conical singularity at the horizon, unless

1. The euclidean time τ and the azimuthal angles ϕ are periodically identified as

$$(\tau, \phi) \equiv (\tau + \beta, \phi - \mathbf{i}\omega\beta)$$

2. The time components of the gauge fields take asymptotic values

$$\mathbf{A}_\tau = -\mathbf{i}\,\mu$$

β, ω, μ **are fixed in terms of M,Q,J for classical black hole**

Interpretation:

Gibbons, Hawking

$\beta = \frac{\partial \mathbf{S_0}}{\partial \mathbf{M}}$ **=inverse temperature,** $\omega = \frac{1}{\beta}\frac{\partial \mathbf{S_0}}{\partial \mathbf{J}}$ **=angular velocity**

$\mu = \frac{1}{\beta}\frac{\partial \mathbf{S_0}}{\partial \mathbf{Q}}$ **=chemical potential**

Scaling from $S_0 \sim \lambda^{D-2}, \quad M, Q \sim \lambda^{D-3}, \quad J \sim \lambda^{D-3}$

$$\beta \sim \frac{\partial S_0}{\partial M} \sim \lambda, \quad \mu \sim \frac{1}{\beta}\frac{\partial S_0}{\partial Q} \sim 1, \quad \omega \sim \frac{1}{\beta}\frac{\partial S_0}{\partial J} \sim \lambda^{-1}$$

In quantum theory we treat β, ω, μ as independent variables, providing boundary condition to the path integral

The gravitational path integral with these boundary conditions gives the grand canonical partition function:

$$Z = \mathrm{Tr}\left[e^{-\beta E - \beta\mu.Q - \beta\omega.J}\right]$$

Consider the gravitational partition function in full space-time geometry with $\beta\omega_0 = 2\pi i$ and $(2J_0)^n$ inserted

$$Z = Tr\left[e^{-\beta H - \mu Q - \beta\omega'.J' - 2\pi i J_0}(2J_0)^n\right] = Tr\left[e^{-\beta H - \mu Q - \beta\omega'.J'}(-1)^F(2J_0)^n\right]$$

Compare this with the index

$$I = e^{S_{BPS} - \beta M_{BPS}} = Tr_{Q,J',k=0}\left[e^{-\beta H}(-1)^F(2J_0)^n\right]$$

In both we sum over J_0, M

In the index I we take the trace for fixed Q, k=0, J' while in Z we sum / integrate over Q, k, J' keeping μ, ω' fixed

Z can be regarded as a sum / integral over Q, k, J' with I as integrand

$$Z = \int d^{n_v}Q \; d^{n'_c}J' \; d^{n_T}k \; e^{\left[S_{BPS} - \beta M_{BPS} - \beta k^2/2M - \beta\omega'.J' - \beta\mu.Q\right]}$$

k: momenta invariant under $\omega'.J'$

$\Rightarrow$ an n_T dimensional space of momenta to integrate over

n'_c: number of generators J'

n_v: number of U(1) gauge fields, i.e. dimension of Q

The contribution to the integral is dominated by the Euclidean black hole saddle point

Gaussian integral around the saddle point produces correction $\propto \ln\lambda$

e.g. k integration gives $\sim (M/\beta)^{n_T/2} \sim e^{\frac{n_T}{2}(D-4)\ln\lambda}$

Q, J′ integrals give $\left(\det\frac{\partial^2 S_{BPS}}{\partial Q^2}\right)^{-1/2}\left(\det\frac{\partial^2 S_{BPS}}{\partial J'^2}\right)^{-1/2} \sim \lambda^{\frac{n_V(D-4)+n'_C(D-2)}{2}}$

Net result:

$$\mathbf{S_{BPS}} = \ln \mathbf{Z} + \beta \mathbf{M_{BPS}} + \beta\omega'.\mathbf{J}' + \beta\mu.\mathbf{Q} + \mathbf{C_E} \ln \lambda$$

with $\mathbf{J}'$, Q evaluated at the saddle, and

$$\mathbf{C_E} = -\frac{1}{2}\left[(\mathbf{n_v} + \mathbf{n_T})(\mathbf{D} - 4) + \mathbf{n}'_\mathbf{c}(\mathbf{D} - 2)\right]$$

$\mathbf{n}'_\mathbf{c}$: number of generators $\mathbf{J}'$

$\mathbf{n_v}$: number of U(1) gauge fields, i.e. dimension of Q

We now need to evaluate the logarithmic correction to ln Z by evaluating the gravitational path integral

Power counting $\Rightarrow$ such contributions come from one loop contribution of massless fields

$\mathbf{K_b}$: Kinetic operator for massless bosonic fields

$\mathbf{K_f}$: Kinetic operator for massless fermionic fields

One loop contribution to Z from massless fields:

$$(\det K_b)^{-1/2}(\det K_f)^{1/2}$$

Correction to ln Z:

$$\delta \ln Z = -\frac{1}{2}\ln\det K_b + \frac{1}{2}\ln\det K_f = -\frac{1}{2}\mathrm{Tr}\ln K_b + \frac{1}{2}\mathrm{Tr}\ln K_f$$

λ dependence arises from $K_b \sim \lambda^{-2}, \qquad K_f \sim \lambda^{-1}$

– can be evaluated using the heat kernel expansion

Seeley; DeWitt; $\cdots$ Vassilevich, hep-th/0306138

Result

$$\delta \ln \mathbf{Z} = \mathbf{C_L} \ln \lambda + \cdots$$

$$\mathbf{C_L} = \int \mathbf{d^4x\, K(x)}$$

K(x) can be computed from the knowledge of K_b and K_f

In N $\geq$ 2 supergravity in D=4, K(x) is proportional to the Gauss-Bonnet term

Charles, Larsen arXiv:1505.01156; Karan, Panda arXiv:2012.12227

$\Rightarrow$ **$C_L \propto$ Euler number**

For D odd, $C_L = 0$

Zero mode contribution:

K_b and / or K_f may have zero eigenvalues arising from broken symmetries like translation, rotation, supersymmetry

– cannot be treated as part of the determinant

1. Remove their contribution from $\delta \ln Z$

e.g. a bosonic mode contributes $(1/\lambda^2)^{-1/2} \sim \lambda$ to Z

$\Rightarrow$ $\ln \lambda$ to $\ln Z$

We need to subtract ($\ln \lambda$) from $\delta \ln Z$ for each bosonic zero mode

Similarly we add ($\ln \lambda$)/2 to $\delta \ln Z$ for each fermionic zero mode

Example: Counting of the number of rotational zero modes n_R

– must be generated by rotation outside the Cartan subalgebra of the group so that it deforms the solution

– must be invariant under $e^{\beta\omega'.J'}$ so that it satisfies the required periodicity as we go around the euclidean time circle.

In D=4 these are rotations about 1 and 2 axis in SU(2) and in D=5 these are rotations about 1 and 2 axes of $SU(2)_R$

$\Rightarrow$ in both D=4 and D=5, $n_R = 2$

Similar analysis can be done for counting the translational zero modes and broken supersymmetry zero modes.

2. We need to find the actual λ dependent contribution to Z from the zero mode integrals

Zero modes typically arise from some broken symmetries

We express the integral over the zero modes as integral over the broken symmetry parameters and carry out the integral

Fermion zero modes are associated with broken supersymmetry

– saturated by the zero mode part of $(2J_3)^n$

The original integration measure, jacobian for the change of variables and the integral over the symmetry transformation parameters can also give ln λ.

Net logarithmic correction from zero modes

$$C_Z \ln\lambda, \qquad C_Z \equiv \frac{1}{2}\, n_T\,(D-4) + \frac{1}{2}\, n_R\,(D-2)$$

Net logarithmic correction to S_{BPS}:

$$(C_E + C_L + C_Z)\ln\lambda = \left[\frac{1}{2}\{(n_R - n'_c)(D-2) - n_V(D-4) + C_L\right] \ln\lambda$$

$$C_L = \int_{\text{Full geometry}} K(x)$$

$n'_c = 0, 1$ in D=4,5, $\quad C_L = 0$ in D=5, $\quad n_R = 2$ in D=4,5.

Final result: Logarithmic correction to the index, computed from the near horizon geometry and the full geometry give the same result$\cdots$

$\cdots$ even though the intermediate steps are quite different

$\Rightarrow$ the index computed from the full geometry correctly reproduces the microscopic results when they are known

e.g. in theories with 16, 32 supersymmetries in D=4, 5

Conclusion

Although this analysis has only reproduced known results, the agreement is significant due to several reasons:

1. The computation using the full geometry uses integration over the same set of modes and same ensemble as that for non-supersymmetric black holes

– gives us confidence in the results for non-supersymmetric black holes for which there is no independent test of the formula

2. In principle, the computation using the full geometry can be used to take into account all configurations that contribute to the index

e.g. multi-centered black holes

3. This formalism may be better suited for exact computation of supersymmetric index from gravitational path integral, e.g. via localization