Emanant Symmetries

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IAS

Meng Cheng and NS, arXiv:2211.12543

NS and Shu-Heng Shao, arXiv:2307.02534

NS and Shu-Heng Shao, to appear.

NS, Sahand Seifnashri, and Shu-Heng Shao,
to appear.

Thanks to Tom Banks

Global symmetry – comparing the UV and the IR

Every internal symmetry operator in the UV is mapped to an internal symmetry operator in the IR (homomorphism)

$$G_{UV} \to G_{IR}$$

Some UV symmetries are trivial in the IR (kernel).

New symmetries in the IR theory (cokernel).

- Emergent/accidental symmetries
  - Arise when the IR theory has no relevant, G_{UV}-preserving, but G_{IR}-violating operators (e.g., $B - L$ in the Standard Model, continuous rotation in lattice models).
  - The low-energy effective Lagrangian includes irrelevant operators that violate the emergent symmetries (e.g., proton decay or neutrino masses in the Standard Model).
- Emanant symmetries emanate from UV space symmetries…

Global symmetry – comparing the UV and the IR

- Emanant symmetries emanate from UV space symmetries, typically from UV translations. Unlike emergent symmetries:
  - There can be relevant operators violating the emanant symmetries, but they are not present in the low-energy effective Lagrangian (or Hamiltonian).
  - The low-energy effective Lagrangian does not include even irrelevant operators that violate the emanant symmetry.
  - The emanant symmetry is exact in the low-energy theory!
  - 't Hooft anomaly matching for emanant symmetries – not for emergent symmetries.
  - Examples (old wine in a new bottle): a system with a $U(1)$ global symmetry with a chemical potential, various spin models, lattice fermions, …

Majorana chain [many references]

A closed lattice with L sites and real periodic fermions χ_ℓ at the sites

$$\chi_\ell = \chi_{\ell+L} \quad , \qquad \{\chi_\ell, \chi_{\ell'}\} = 2\delta_{\ell,\ell'}$$

Impose invariance under lattice translation ($\ell \to \ell + 1$) and fermion-parity ($\chi_\ell \to -\chi_\ell$)

Typical Hamiltonian $\quad H_+ = \frac{i}{2}\sum_{\ell=1}^{L} \chi_{\ell+1}\chi_\ell$

Add a fermion-parity defect (equivalently, use H_+ with anti-periodic boundary conditions). $\quad H_- = \frac{i}{2}\sum_{\ell=1}^{L-1} \chi_{\ell+1}\chi_\ell - \frac{i}{2}\chi_1\chi_L$

Most of our discussion is independent of the details of $H_\pm$.

Four fermionic theories:

- Even L. H_- leads in the continuum to the NSNS Majorana CFT and H_+ leads to the RR theory.
- Odd L. H_- leads in the continuum to the RNS theory Majorana CFT and H_+ leads to the NSR theory.

Majorana chain – even $L = 2N$ [many references]

Typical Hamiltonians

$$H_\pm = \frac{i}{2}\sum_{\ell=1}^{L-1} \chi_{\ell+1}\chi_\ell \pm \frac{i}{2}\chi_1\chi_L$$

Symmetries generated by translation $T_\pm$ and fermion parity $(-1)^F$

For H_-

$$T_-^L = (-1)^F$$
$$T_-(-1)^F = (-1)^F T_-$$

For H_+

$$T_+^L = 1$$
$$T_+(-1)^F = -(-1)^F T_+$$

[Rahmani, Zhu, Franz, Affleck; Hsieh, Hal´asz, Grover]

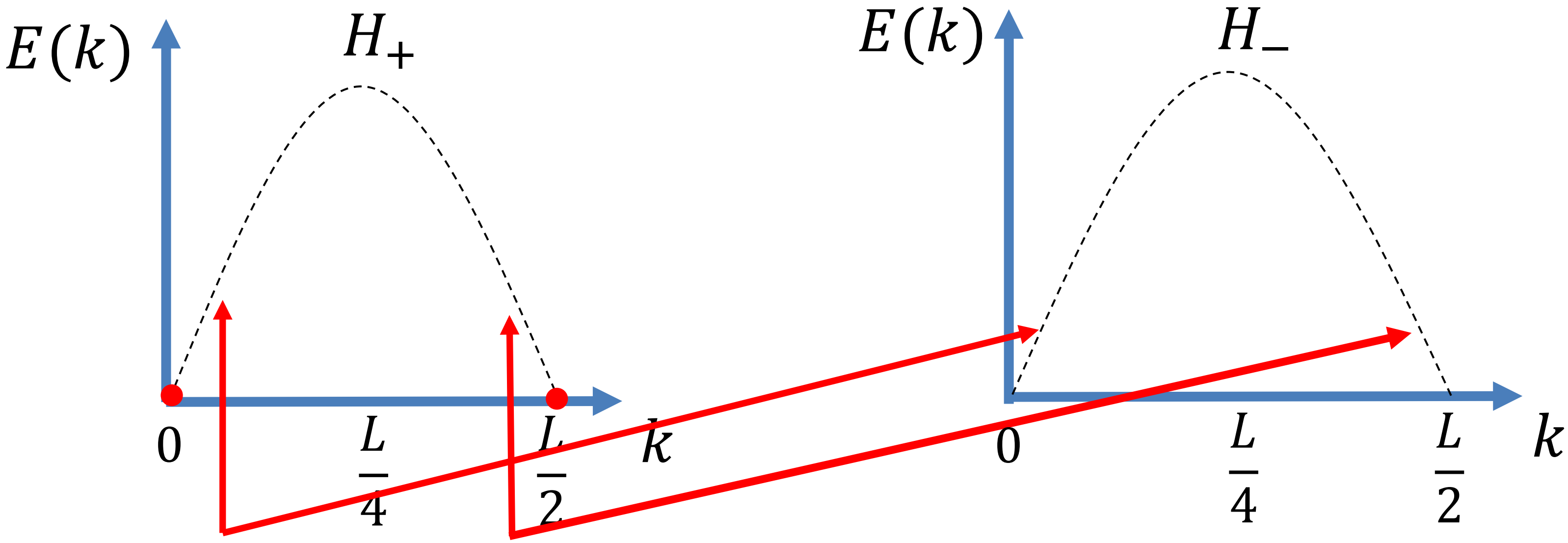
The minus sign reflects an anomaly between fermion-parity and lattice-translation.

In the continuum, no anomaly involving translations. How is this UV anomaly realized at low energies?

Majorana chain – even $L = 2N$ [many references]

$$H_{\pm} = \frac{i}{2}\sum_{\ell=1}^{L-1}\chi_{\ell+1}\chi_{\ell} \pm \frac{i}{2}\chi_1\chi_L$$

For the specific $H_{\pm}$, normal mode expansion:



- Right-movers and left-movers from the two ends of the spectrum
- H_+ leads to the RR theory. H_- leads to the NSNS theory.
- On the lattice, only $(-1)^F$; no $(-1)^{F_L}$, $(-1)^{F_R}$.
- Without a chiral symmetry, why is the fermion massless?

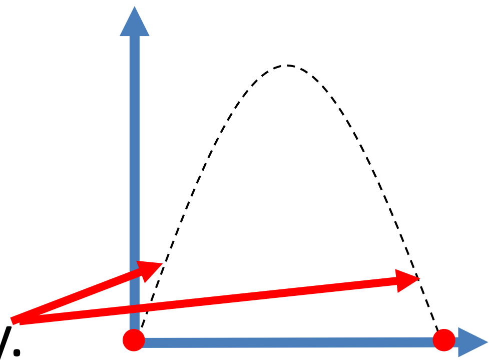
Majorana chain – even $L = 2N$

Consider H_+. On the lattice, no $(-1)^{F_L}$. In the IR, it emanates from T_+.

$$T_+^L = 1$$

$$T_+ = (-1)^{F_L} e^{\frac{2\pi i P_+}{L}}$$

$$e^{2\pi i P_+} = 1$$



- P_+ is the momentum of the continuum RR theory.
- On the lattice, only T_+ is well-defined. In the continuum, $(-1)^{F_L}$ and P_+ are separately meaningful exact symmetries.
- The relation $T_+ = (-1)^{F_L} e^{\frac{2\pi i P_+}{L}}$ is exact, without finite L corrections.
- The anomaly in the continuum RR theory [...; Delmastro, Gaiotto, Gomis; ...]

$$(-1)^F (-1)^{F_L} = -(-1)^{F_L} (-1)^F$$

matches the UV fermion-parity/lattice-translation anomaly.

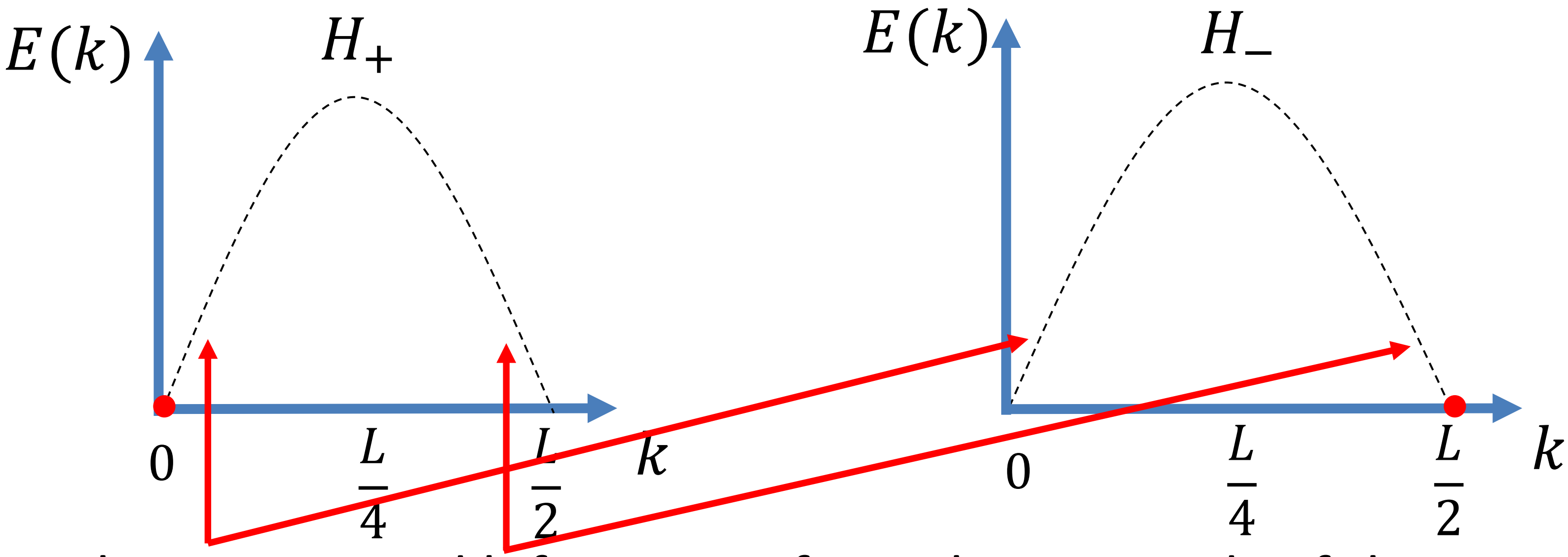
Similarly for H_-, except $e^{2\pi i\, P_-} = T_-^L = (-1)^F$

Majorana chain – odd $L = 2N + 1$

$$H_{\pm} = \frac{i}{2}\sum_{\ell=1}^{L-1} \chi_{\ell+1}\chi_{\ell} \pm \frac{i}{2}\chi_1\chi_L$$

No $(-1)^{F_L}$, $(-1)^{F_R}$, $(-1)^{F}$.

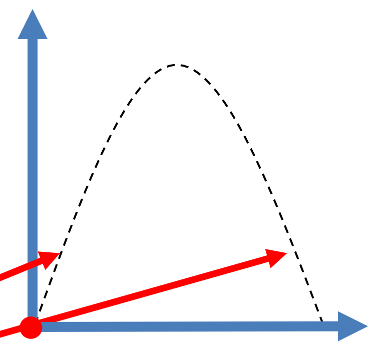
Only lattice translation $T_{\pm}$, with an anomaly $T_{\pm}^{L} = e^{\mp\frac{2\pi i}{16}}$



- Right-movers and left-movers from the two ends of the spectrum
- H_+ leads to the NSR theory. H_- leads to the RNS theory.

Majorana chain – odd $L = 2N + 1$

- No $(-1)^{F_L}, (-1)^{F_R}, (-1)^F$ on the lattice.
- Consider H_+. In the IR, $(-1)^{F_L}$ emanates from T_+



$$T_+^L = e^{-\frac{2\pi i}{16}}$$

$$T_+ = (-1)^{F_L} e^{\frac{2\pi i P_+}{L}}$$

$$e^{2\pi i P_+} = (-1)^{F_L} e^{-\frac{2\pi i}{16}}$$

  - P_+ is the momentum of the continuum NSR theory.
  - On the lattice, only T_+ is well-defined. In the continuum, $(-1)^{F_L}$ and P_+ are separately meaningful exact symmetries.
  - The relation $T_+ = (-1)^{F_L} e^{\frac{2\pi i P_+}{L}}$ is exact, without finite L corrections.
- For H_-: $+ \to -$, $F_L \to F_R$, and we find the RNS theory.

From the Majorana chain to the Ising model – GSO on the lattice

Sum over the "spin structures" by first doubling the Hilbert space (related work in [Baake, Chaselon, Schlottmann; Grimm, Schutz; Grimm])

$$\widetilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$$

with the Hamiltonian $$\widetilde{H} = \begin{pmatrix} H_- & 0 \\ 0 & H_+ \end{pmatrix}$$

(H_+ corresponds to fermions with periodic boundary conditions. H_- corresponds to fermions with antiperiodic boundary conditions.)

Translation symmetry $$\tilde{T} = \begin{pmatrix} T_- & 0 \\ 0 & T_+ \end{pmatrix}$$

Because of the doubling of the Hilbert space, a quantum $\mathbb{Z}_2$ symmetry

$$\tilde{\eta} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

From the Majorana chain to the Ising model – even $L = 2N$

Some operators in the doubled Hilbert space $\widetilde{\mathcal{H}}$ are nonlocal. So imitating the continuum, we project:

$\tilde{\eta}(-1)^F = +1$ leads to the Ising model $\widetilde{\mathcal{H}}|_{Ising} = \mathcal{H}_{Ising}$

Using a Jordan-Wigner transformation in $\mathcal{H}_{Ising}$,

$$H_{Ising} = \widetilde{H}\Big|_{Ising} = -\frac{1}{2}\sum_{j=1}^{N} Z_j - \frac{1}{2}\sum_{j=1}^{N} X_j X_{j+1}$$

(X_j, Y_j, Z_j are Pauli matrices at the site $j = 1, \cdots, N$)

Similarly, $\tilde{\eta}(-1)^F = -1$ leads to the $\mathbb{Z}_2$-twisted Ising model

$$H_{twisted\ Ising} = -\frac{1}{2}\sum_{j=1}^{N} Z_j - \frac{1}{2}\sum_{j=1}^{N-1} X_j X_{j+1} + \frac{1}{2} X_N X_1$$

From the Majorana chain to the Ising model – even $L = 2N$

$\tilde{T} = \begin{pmatrix} T_- & 0 \\ 0 & T_+ \end{pmatrix}$ does not act in $\widetilde{\mathcal{H}}|_{Ising}$. It is not a symmetry.

$\tilde{T}^2$ and $\tilde{\eta}$ act in $\widetilde{\mathcal{H}}|_{Ising}$. Standard symmetries of the Ising model

$$T_{Ising} = \tilde{T}^2\Big|_{Ising}, \qquad \eta = \tilde{\eta}\Big|_{Ising}$$

Lattice-translation $\quad T_{Ising}^N = 1$

$\mathbb{Z}_2$ Ising symmetry $\quad \eta^2 = 1$

$\begin{pmatrix} T_- & 0 \\ 0 & 0 \end{pmatrix}$ commutes with the $\tilde{\eta}(-1)^F = +1$ projection and hence acts in $\widetilde{\mathcal{H}}|_{Ising}$.

$D = \begin{pmatrix} T_- & 0 \\ 0 & 0 \end{pmatrix}|_{Ising}$ is a new symmetry of the lattice Ising model.

From the Majorana chain to the Ising model – even $L = 2N$

New noninvertible symmetry of the lattice Ising model

$$D = \begin{pmatrix} T_- & 0 \\ 0 & 0 \end{pmatrix}\Bigg|_{Ising}$$

$$D^2 = \frac{1}{2}(1+\eta)T_{Ising}$$

Can express D in terms of the local operators X_j, Y_j, Z_j.

From the Majorana chain to the Ising model – even $L = 2N$

The noninvertible lattice symmetry $D = \begin{pmatrix} T_- & 0 \\ 0 & 0 \end{pmatrix}|_{Ising}$ flows to a noninvertible symmetry of the continuum theory $\mathcal{D}$ [Oshikawa, Affleck; Petkova, Zuber; Frohlich, Fuchs, Runkel, Schweigert; Chang, Lin, Shao, Wang, Yin]

$$D = \frac{1}{\sqrt{2}} \mathcal{D} e^{\frac{2\pi i P}{2N}}$$

$$\mathcal{D}^2 = 1 + \eta \quad , \quad \eta^2 = 1 \quad , \quad \eta\mathcal{D} = \mathcal{D}\eta = \mathcal{D} \quad , \quad e^{2\pi i P} = 1$$

D and $\mathcal{D}$ satisfy different algebras, $D^2 = \frac{1}{2}(1+\eta)T_{Ising}$.

$\mathcal{D}$ is an emanant noninvertible symmetry. It is exact in the IR effective theory. (Not violated even by irrelevant operators.)

On the lattice, only D and T_{Ising}. In the continuum, P and $\mathcal{D}$.

The relation $D = \frac{1}{\sqrt{2}} \mathcal{D} e^{\frac{2\pi i P}{2N}}$ is exact. No finite N corrections.

From the Majorana chain to the Ising model – odd $L = 2N + 1$

In this case, no projection is needed.

A Jordan-Wigner transformation in the doubled Hilbert space $\widetilde{\mathcal{H}}$ leads to the Ising model with a D defect [Schutz; Grimm, Schutz; Grimm; Ho, Cincio, Moradi, Gaiotto, Vidal; Hauru, Evenbly, Ho, Gaiotto, Vidal; Aasen, Mong, Fendley]

$$H = -\frac{1}{2}\sum_{j=1}^{N} Z_j - \frac{1}{2}\sum_{j=1}^{N} X_j X_{j+1} - \frac{1}{2} X_1 Y_{N+1}$$

It flows in the IR to the Ising CFT with a noninvertible defect $\mathcal{D}$ [Oshikawa, Affleck; Petkova, Zuber; Frohlich, Fuchs, Runkel, Schweigert; Chang, Lin, Shao, Wang, Yin].

Summary

- UV-translation can lead to an emanant internal symmetry. Unlike an emergent/accidental symmetry, it is exact at low energies – not violated by relevant or irrelevant operators.
- Anomalies involving UV-translations are matched by anomalies in emanant symmetries.
- Four versions of the lattice Majorana chain flow to the continuum Majorana theory with four different defects, NSNS, RR, NSR, and RNS. In each case, a chiral fermion parity symmetry emanates from lattice-translation T. It is exact in the low-energy theory.
- Summing over the lattice spin structures leads to three bosonic lattice models: Ising, $\mathbb{Z}_2$-twisted Ising, and Ising with a D defect.
- D is an exact noninvertible symmetry of the lattice model.
- These lattice models flow to the three continuum Ising CFTs with defects (corresponding to $1, \epsilon, \sigma$).
- The noninvertible duality symmetry $\mathcal{D}$ of the CFT emanates from D.

Thank you