Off Shell Strings & Black Hole Entropy

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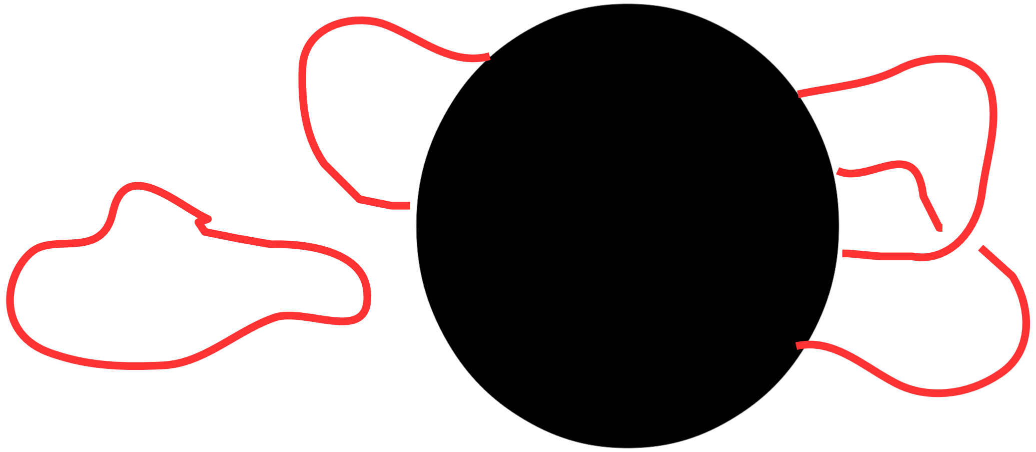
Last time I spoke at Strings (Brussels 2019), I reviewed recent progress in horizon thermodynamics. My last slide included the following image:



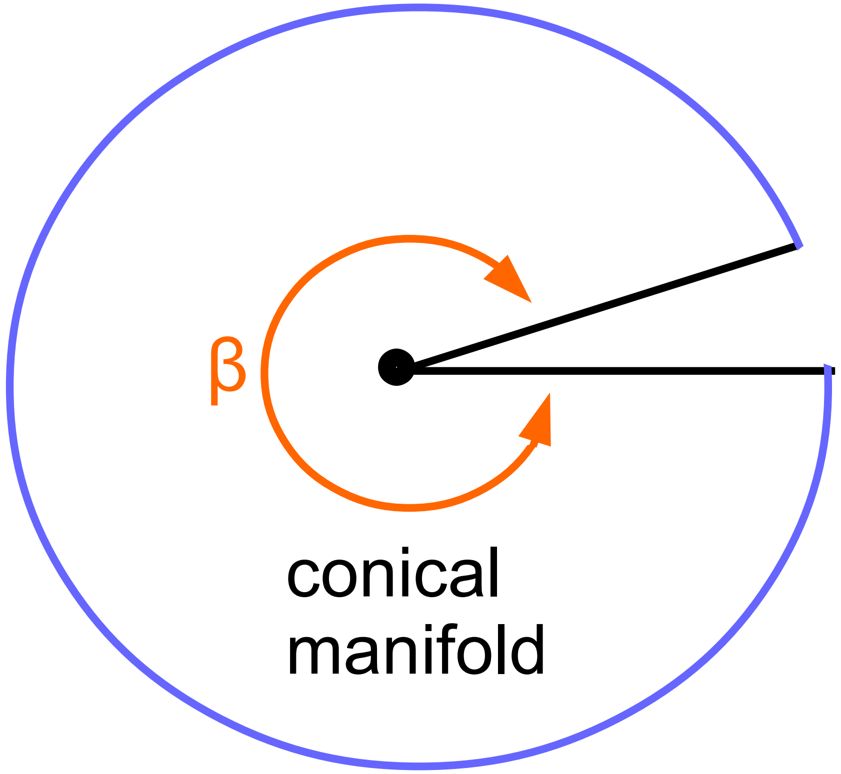
Susskind & Uglum argued that the Bekenstein-Hawking entropy comes mainly from strings that cross the horizon, but their calculation of A/4 requires off-shell string theory

At the time I wrote this, to me off-shell string theory meant

“scary stuff I don't understand”.

No longer true and my goal is to communicate how this works...

Susskind & Uglum (1994) calculation of BH entropy

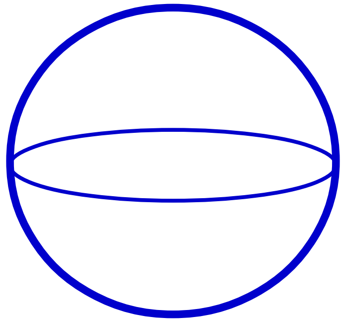


calculated for infinite mass limit of Schwarzschild, i.e. Rindler

S&U discuss D = 10 superstrings,
but D = 26 bosonic case very similar.

$$\boxed{S = (1 - \beta\partial_\beta)Z_0 \propto \frac{A}{4G}}$$

off-shell sphere
worldsheet amplitude



Einstein-Hilbert contribution from tip of cone when $\beta \neq 2\pi$.

exponentiate this to get target space partition function (any # of spheres)
Hence target space action: $-I_0 = Z_0$

While widely cited for other reasons, the vast majority of the community did not understand this argument, since the substantive string theory content was implicit (key equation comes from 1 Tseytlin cite).


SUSSKIND & UGLUM
TSEYTLIN'S OFF-SHELL FORMALISM

Classical String Action and Sphere Diagrams

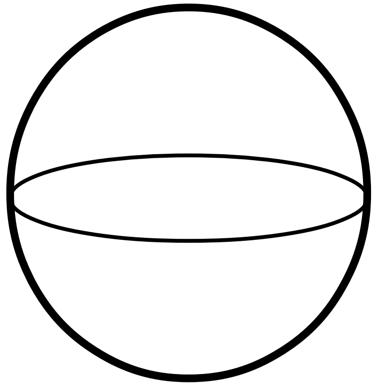
Leading order classical (Euclidean) bosonic string theory action (up to total derivative):

$$I_0 = -\int \mathrm{d}^D X \sqrt{G} e^{-2\Phi} \left[4\nabla^2 \Phi + R - \frac{1}{12} H_{\mu\nu\xi} H^{\mu\nu\xi} + O(\alpha') \right]$$

In on-shell formalism, derived *indirectly* through $\beta = 0 \longrightarrow$ EOM.

Suppose we try to get it directly from the worldsheet?

As $e^{-2\Phi} \sim 1/g_s^2$, this should come from *sphere diagrams* evaluated in a string background (i.e. CFT with c = 0) corresponding to the target space fields.



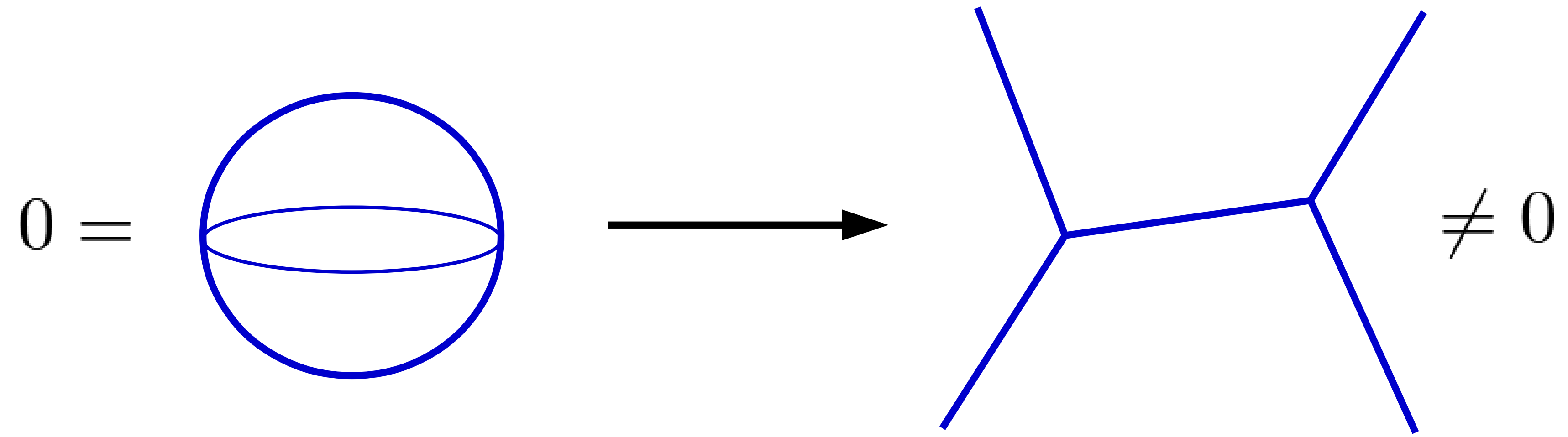
But sphere diagrams with 0 insertions vanish for any string CFT:

$$1/|SL(2,C)| = 1/\infty = 0$$

Corresponds to fact that I_0 vanishes on-shell,
(up to a boundary term) due to EOM of dilaton Φ.

Why is there a tree level S-matrix?

However, as we all know the tree-level S-matrix does *not* vanish when perturbing around a CFT:



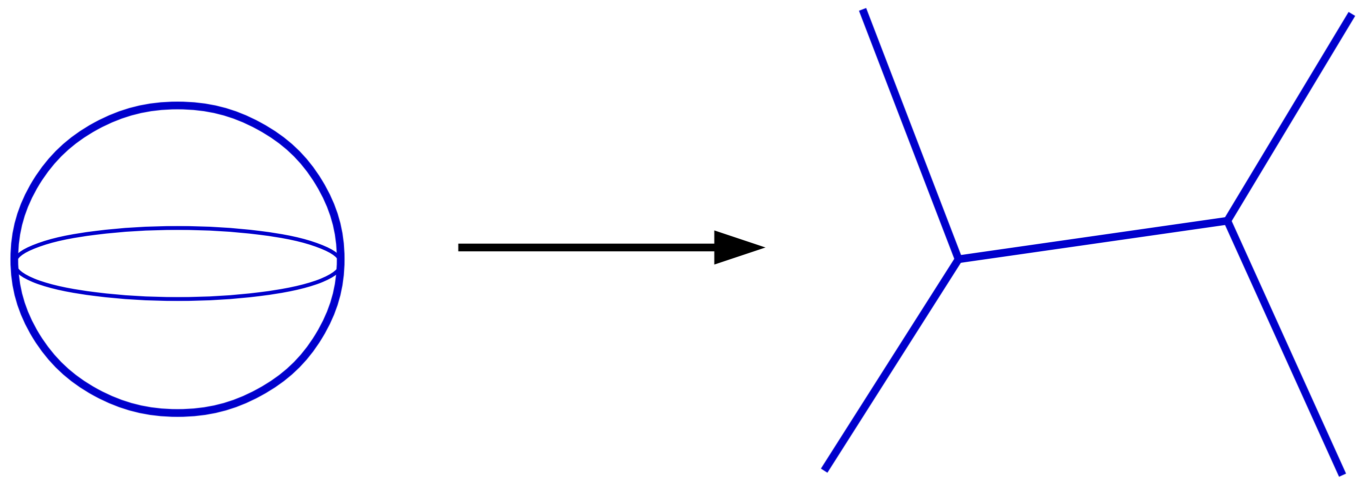
Easy enough to see in usual formalism (insert vertex operators, fix 3 points).

But suppose we think of each external line as coming from a small amplitude coherent wave proportional to $e^{iP_\mu X^\mu}$, (+/- freq. corresponds to in/out states) & we evaluate the contribution to the effective action e^{-I_0} in this *new* background, then we extract S-matrix elements from product of fields.

Relative to the new background there are 0 insertions. So how do we get a nonzero answer?

2 Resolutions of the S-matrix Puzzle

$0 =$



$\neq 0$

There are 2 distinct resolutions to this puzzle, which are consistent (give same amplitude)

1) The new background is actually ***off-shell*** and so the action doesn't vanish. For *n*-string amplitude, any *n-1* plane waves induce an $O(\varphi^{n-1})$ beta function in *n*-th plane wave.

2) Or, we can do a small $O(\varphi^{n-1})$ adjustment to the fields to cancel this beta fn, to ensure an **on-shell** background. But the "tails" of this adjustment fall off slowly in IR so you have to worry about boundary terms in the on-shell action!

A 1st order perturbation to a solution doesn't change log Z, so these agree!
(Don't need to know what these bdy terms are to make this argument)

Why there are long tails in on-shell scattering processes

Recall that the dimension of an operator in string theory, (i.e.its *linear* beta fn), is related to the mass-shell condition by:

$$\Delta - 2 \propto P^2 + M^2$$

An on-shell external line corresponds to a coupling which is **marginal** at linear order, and hence satisfies the mass shell condition:

$$P^2 + M^2 = 0$$

Such operators lead to CFTs at linear order, but interactions also give nonlinear beta functions. Suppose that their effect is to introduce a perturbation of the form $\beta^{\varphi}(P_\mu)$ of some field φ

Then we can cancel the nonlinear beta fn with an linear one, by adjusting the oscillators of the field by

$$\varphi \sim \frac{1}{\Delta - 2}\beta^{\varphi}(P_\mu)$$

The pole at $\Delta = 2$ causes a “long tail” at infty (inverse laplacian in position space)

On-Shell vs. Off-shell Black Hole Entropy

These 2 resolutions are reminiscent of GH calculations of the BH entropy by varying free energy:

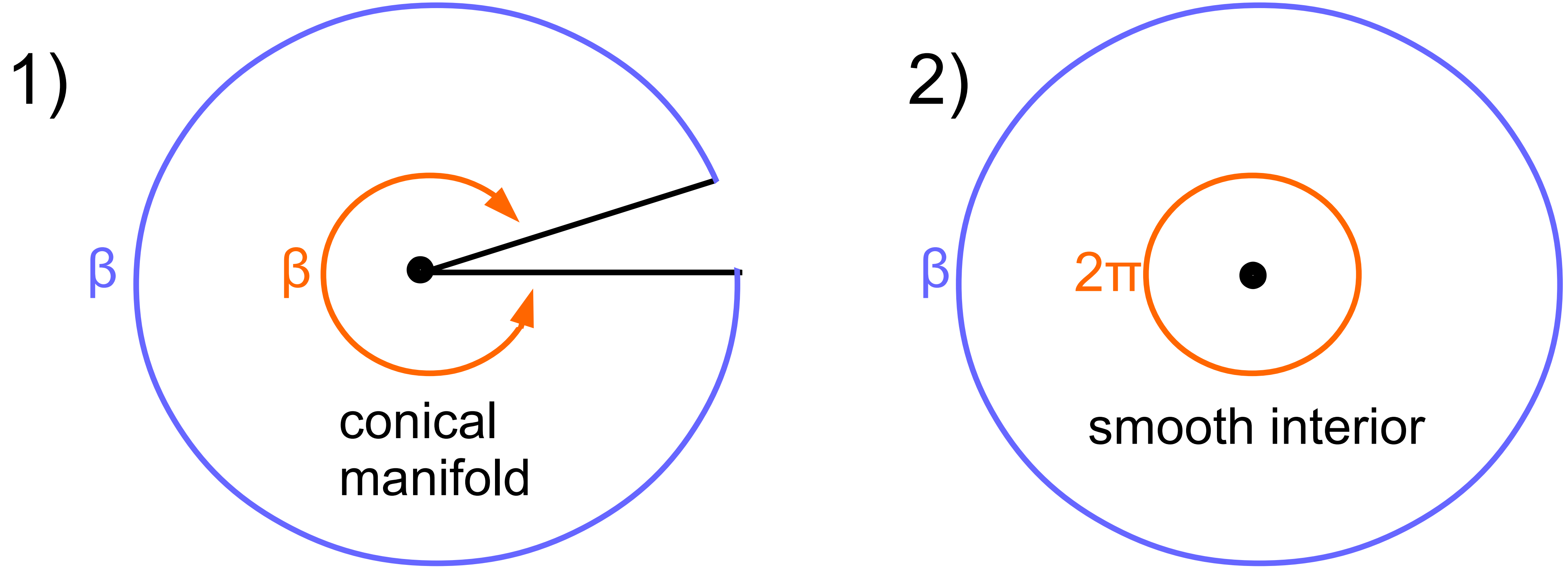
$$S = (1 - \beta\partial_\beta)\log Z$$

Here it is possible to calculate BH entropy either:

1) **off-shell,** by introducing a conical singularity of angle $\beta \neq 2\pi$ at the horizon (boundary terms drop out as they are linear in β .) OR

2) **on-shell**, by changing the boundary conditions at spatial infty so that $\beta \neq 2\pi$ at asymptotic infty, and solving for the gravitational saddle in the interior.

Both give same answer, again because 1st order perturbation to solution doesn't change the eff. action!



It follows that in order to derive the classical string action properly, we are going to need to understand at least *one* of the following from a worldsheet perspective:

TARGET SPACE BOUNDARY TERMS

OFF SHELL STRING THEORY

One way to think about the utility of going off-shell, is that it avoids the need for boundary terms!

In the remainder of this talk I will explain how to take string theory off-shell using Tesytlin's NLSM formalism, which Amr and I explained, justified, and extended in these papers:

Off-Shell Strings I: S-matrix and Action

Off-Shell Strings II: Black Hole Entropy

Primarily reviewing Tseytlin 86, 86, 87, 87, 89, 91, 07; Andreev-Metsaev-T. 90, but also many other papers!

Also important to us: Callan-Friedan-Martinec-Perry 85, Fridling-van de Ven 86, Jevicki-Lee 88, Brustein-Yankielowicz-Nemeschansky 87 Shore 87, de Alwis 89, Liu-Polchinski 88, Seiberg 87, Curci-Paffuti 86.

Related early work by Polyakov, Friedan, Lovelace, Sen, Hull, Witten + several others.]

Off-Shell Worldsheet QFT

Study QFT using conformal perturbation theory around a string CFT (c = 0).

Perturbation to Lagrangian is still required to be scalars (Lorentz Invariance) but no longer have to be marginal (allow $\Delta \neq 2$) or even primary (allow couplings to 2d Ricci):

$$\mathcal{L}_{\text{pert.}} = \sum_i \left[\epsilon^{\Delta_i - 2} \phi^i \mathcal{P}_i \; + \; \epsilon^{\Delta_{Ri} - 2} \tilde{\Phi}^i \mathcal{P}_i R \right]$$

propagating
target space mode

primary
operator

“dilatonic” mode
(does not propagate)

primary times
2d Ricci scalar

(We do not allow descendents or higher powers of R, as their Δ is too irrelevant to be in regime of validity which will be given later. Total derivative terms ignored as these are pure gauge modes.)

The “dilatonic” field $\tilde{\Phi}$ includes a nonprimary piece of the metric $G_{\mu\nu}$ as well. Does not propagate (constraints). The propagating dilaton is a *different* linear combination of Φ and $G_{\mu\nu}$ which *is* a primary (cf. Polchinski).

ϵ is the UV cutoff of the theory, e.g. a proper distance cutoff on the sphere, which prevents two operator insertions from coming arbitrarily close to each other. Included in Lagrangian to ensure ϵ is only dimensionful constant.

Treatment of Weyl Frame

Unlike *noncritical* string theory, we do not integrate over the Weyl mode $\sqrt{g}$ as a dynamical field in the worldsheet path integral, because that would change # d.o.f...

(Need continuity as QFT $\longrightarrow$ CFT.)

Instead we fix the Weyl factor *arbitrarily*, which is OK because it can be shown that different values of the Weyl frame $\sqrt{g}$ correspond to field redefs in target space (next slide) and thus the different choices are physically equivalent.

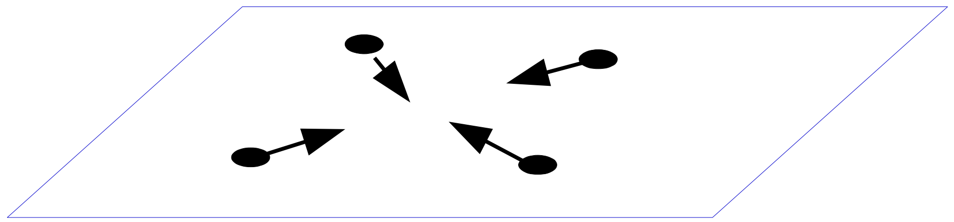
We still gauge fix the 2d diffeomorphism symmetry, resulting in a *b,c* ghost sector which turns out to be the *same* as in the usual on-shell covariant formalism.

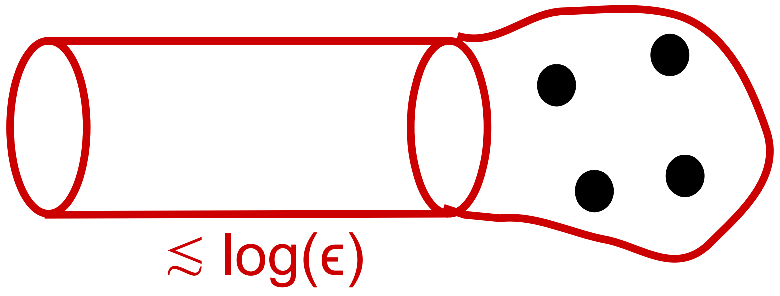
(More details given in our paper.)

Role of UV cutoff

UV cutoff ϵ needed because worldsheet theory is now QFT, not CFT. As pointed out by Susskind 94, strings wildly fluctuate in UV. Without a cutoff, a given string will fill *all of spacetime.*

UV cutoff on worldsheet $\longleftrightarrow$ IR cutoff on target space propagators





Varying ϵ is related to changing the Weyl frame on the worldsheet, which is RG flow. RG corresponds to a field redefinition of target space fields, and hence different values of ϵ are physically equivalent. I.e. we can adjust the value of a local field to account for coherent strings coming in from elsewhere. 3 interesting regimes on unit sphere:

$\log \epsilon = O(1)$: Approximately local target space action (>> string length)

$\epsilon \to 0$: Euclidean S-matrix

different epsilon!

$\log \epsilon \to i\infty$: Lorentzian S-matrix, gives $i\varepsilon$-prescription (ε needed to damp limit, as in Witten '15)

SPURIOUS TADPOLES

In general, 1-point function insertions in a *compact* CFT vanish by conformal invariance.
This is good, as it seems to imply that we will automatically find that the sphere action is stationary around a string CFT! *(Noncompact case: OK to have nonzero n = 1 fn for non-normalizable modes.)*

But there is a problem. There is a special class of “spurious tadpoles” which are terms on worldsheet that are powers of *R* alone (no *X* dependence). Thse have a nonvanishing 1 point function:

$\int d^2\, 1 \longrightarrow$ 2d cosmological constant, i.e. bosonic tachyon zero mode

$\int d^2\, R \longrightarrow$ 2d Einstein-Hilbert, i.e. dilaton zero mode

$\int d^2\, R^p,\ \ p \geq 2 \longrightarrow$ does not appear in string spectrum & we don't know how to deal with it (hence cannot allow in RG flow either)

working with Amr A. & Alex Frenkel to try to fix this problem!

(FIRST / SECOND) TSEYTLIN PRESCRIPTIONS

To eliminate these unwanted tadpoles, Tseytlin does a strange thing: instead of fixing 3 points he *differentiates* the sphere QFT partition function with *n* integrated vertex operators wrt log of UV cutoff ϵ :

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off-shell tree amplitude

sphere QFT partition fn with all n insertions *integrated* over worldsheet

T1: $$\boxed{A_{0,n} = \frac{\partial}{\partial \log \epsilon} K_{0,n}}$$

This prescription eliminates the spurious dilaton tadpole.

Regime of validity: $2(1 - 1/n) < \Delta < 2(1 + 1/n)$ [i.e. near marginal]

log div. of CC (Δ = 0)

log div. of R^2 (Δ = 4)

T2: $$\boxed{A_{0,n} = \left(\frac{\partial}{\partial \log \epsilon} + \frac{1}{2} \frac{\partial^2}{(\partial \log \epsilon)^2} \right) K_{0,n}}$$

Also removes spurious bosonic tachyon tadpole. (Not needed for superstrings).

Regime of validity: $0 < \Delta < 2(1 + 1/n)$ [relevant through slightly irrelevant]

unitarity bound

log div. of R^2 (Δ = 4)

2 point amplitude

i) 2-pt primary amplitude:

$$K_{0,2} = \int_{S^2} \langle \mathcal{P}(0)\mathcal{P}(z)\rangle \propto \frac{1}{1-\Delta}$$

pole comes from log div. @ $\Delta = 1$

But after applying **T2** we obtain: $\boxed{A_{0,2} \propto 2-\Delta}$ (**T1** similar near $\Delta = 2$)

c-function like, (+/-) for relevant / irrelevant def.
cf. Tseytlin 87, 07.

ii) 2-pt "dilatonic" coupling:

$$K_{0,2}^{\mathrm{non-pr}} = \int_{S^2} R^2 \langle \mathcal{P}(0)\mathcal{P}(z)\rangle \propto \frac{1}{3-\Delta}$$

$$\boxed{A_{0,2} \propto \frac{(\Delta-1)(\Delta-2)}{3-\Delta}}$$

this expression looks strange at first, but note that when $2 < \Delta < 3$ (in regime of validity) this has the *opposite sign* relative to the primary case.

$\Rightarrow$ replicates CONFORMAL MODE "PROBLEM" of Euclid. GR: $\sqrt{G}$ modes have (–) norm.

If we had gotten the same sign, would've been impossible to recover low energy gravity action!

3+ point amplitudes

Want solution to EOM *if and only if* CFT: $\beta^i = 0 \iff E_i = 0$

We showed that, to all orders in perturbation theory in *n*, the action gives acceptable EOM so long as it works for *marginal primaries:*

Because: i) to check above statement, we only care about value at leading nonzero order in *n*

ii) beta functions for dilatonic terms are always +1 higher order in *n* than the beta functions for primaries. (Curci-Pafutti, an example of a WZ consistency relation.)

But for *marginal primaries*, we just need to know that **T1** or **T2** recover the ususal on-shell S-matrix.

There is a beautiful story for why this happens, which involves gauge orbits of the SL(2,C) CKG (next slide).

Thus, Tseytlin's prescriptions give the correct tree-level EOM for string theory:

- to all orders in g_s (i.e. at arbitrary *n*)
- and a bit beyond all orders in α' (i.e. a $\Delta - 2$ expansion)

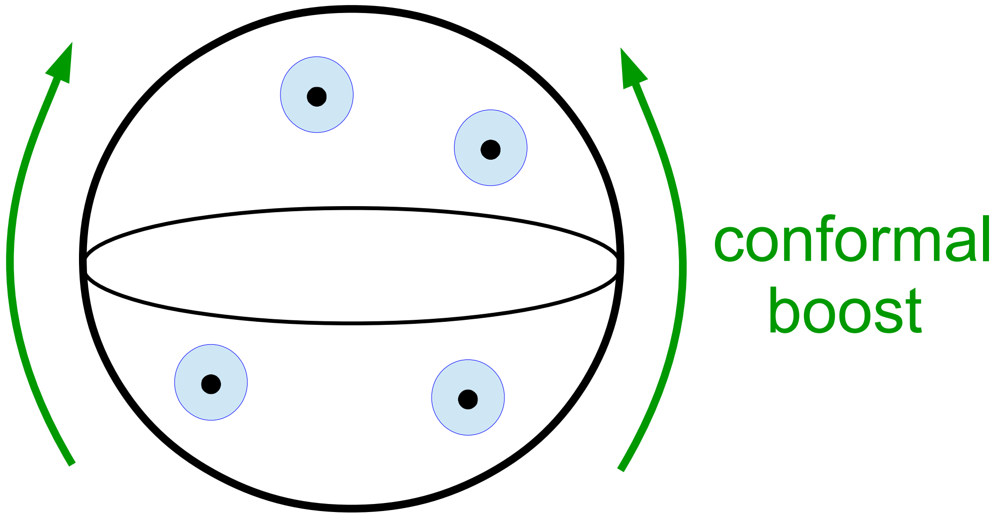
Gauge Orbits of SL(2,C)

Consider a sphere with *n* marginal primary vertex operators.

By conformal symmetry, the contributions to the amplitude are SL(2,C) invariant, and we want to quotient this out.

Let us regulate by forbidding vertex operators to come closer than proper distance ϵ on S^2.

This cuts off the size of the SL(2,C) gauge orbit Ω, and gives it a finite Haar volume. The volume involves an integral over the 3d hyperbolic space of celestial boosts:



4 vertex operators on sphere, surrounded by ϵ sized disks.

For $n \geq 3$, a sufficient boost of the vertex operators in any direction eventually causes 2 disks of size ϵ to intersect.

$$H_3 = \frac{\mathrm{SL}(2,\mathrm{C})}{\mathrm{SU}(2)}$$

rotations preserve cutoff so only give overall multiplicative factor

Gauge Orbits of SL(2,C)

For small ϵ the volume of the gauge orbit is given by an asymptotic expansion:

$$\mathrm{Vol}(\Omega) = a\epsilon^{-2} + \frac{4\pi}{2}\log(\epsilon) + b + O(\epsilon^2),$$

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(as usual for logs)

The coefficients *a*, *b* are SCHEME DEPENDENT and depend in an exceedingly complicated way on *n,* and the conformal cross-ratios when n > 3. If we could calculate these coefficients explcitly, we could simply divide by Vol(Ω) *inside* the path integral. But no sane person would want to do this calculation.

By differentiating wrt log ϵ (and throwing out the leading power), this is equivalent to quotienting out each gauge orbit Ω by assigning it a constant factor (regardless of its size).

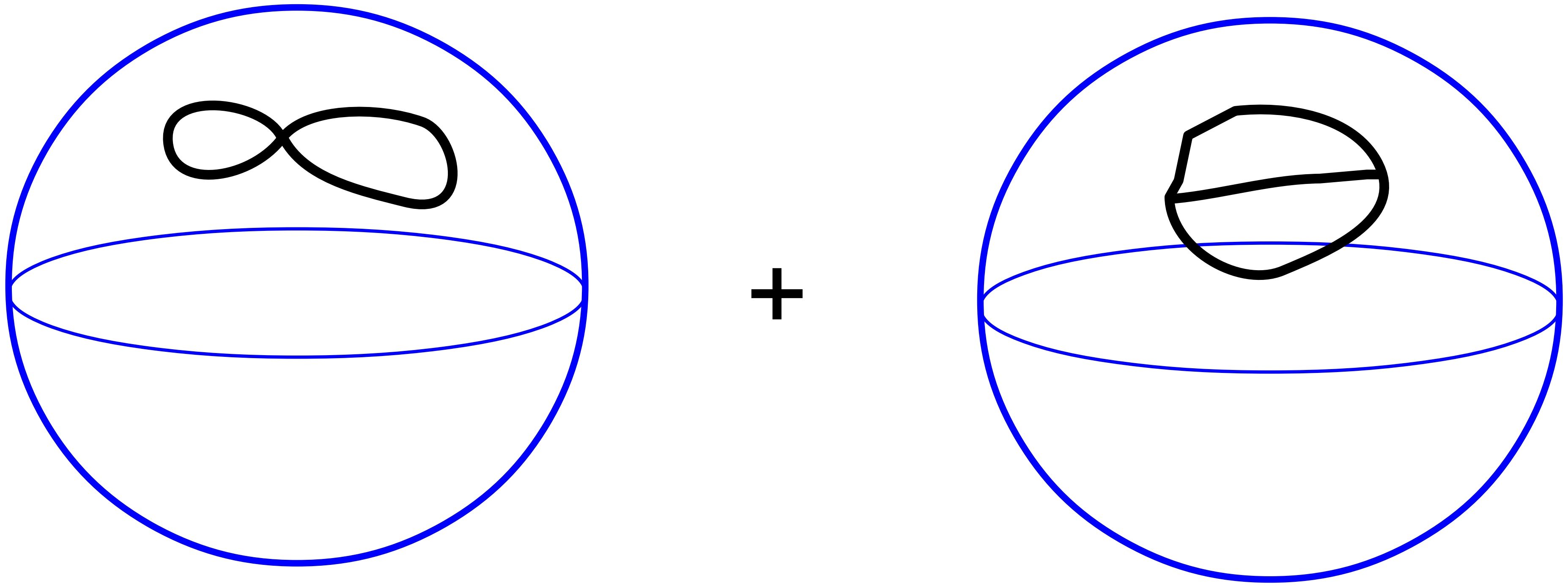
T2 automatically throws out the leading order power (the 2d C.C.)

T1 also works if you throw out powers by hand (always allowed, as this is a valid RG scheme)

(Similar story for open string disk with CKG = SL(2,R) but no log in that case, so O(1) term is universal.)

Calculation of leading order action (following Tseytlin very closely)

Comes from 2-loop Feynan diagrams of *nonzero* modes of X on spherical worldsheet. (But NLSM tricks reduce to 1-loop!)
Have to be *very* careful about i) measure factors, ii) separation between zero modes & nonzero modes.

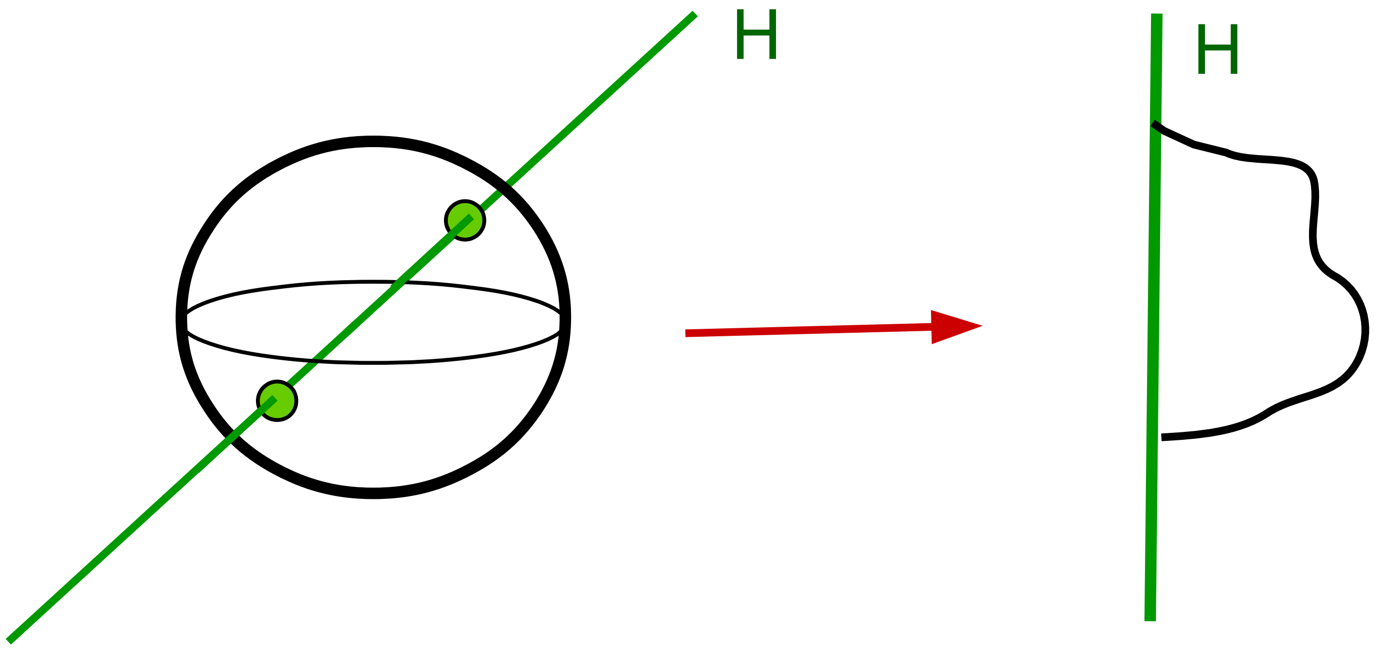


$$I_0 = -\int \overbrace{\mathrm{d}^D X}^{\text{path integral over zero mode}} \underbrace{\sqrt{G}e^{-2\Phi}}_{\substack{\text{generalized volume} \\ \text{(= sphere Z in Tseytlin's RG scheme)}}} \overbrace{\left[4\nabla^2\Phi + R - \frac{1}{12}H_{\mu\nu\xi}H^{\mu\nu\xi} + O(\alpha')\right]}^{\text{comes from applying } \textbf{T1} \text{ or } \textbf{T2} \text{ prescription to log divergence of nonzero modes}}$$

target space EH, leads to S&U BH entropy
the R log(ϵ) term comes from “missing” zero mode

The "Open String" Picture

Susskind & Uglum also argued that spheres crossing a Rindler horizon could be viewed as an entanglement entropy arising from strings crossing the horizon H ("open string" edge modes, from the perspective of an observer restricted to one side of H):



Some issues with this picture, if we try to go beyond the level of "cartoons":

i) 2d EH term gives $1/g_s$ extra factor per string endpoint, no manifest statistical interpretation (hence not really induced gravity—need to induce EH term on worldsheet!)

ii) Strings fluctuate wildly enough so they always cross H infinitely often!

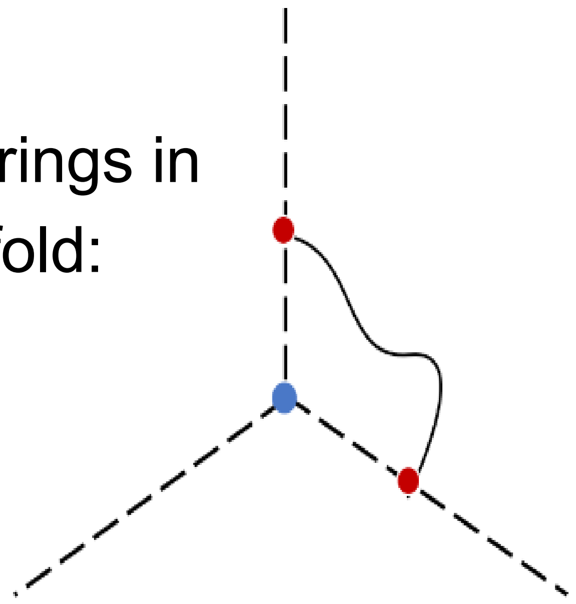
iii) Need acceptable boundary conditions on endpoint.

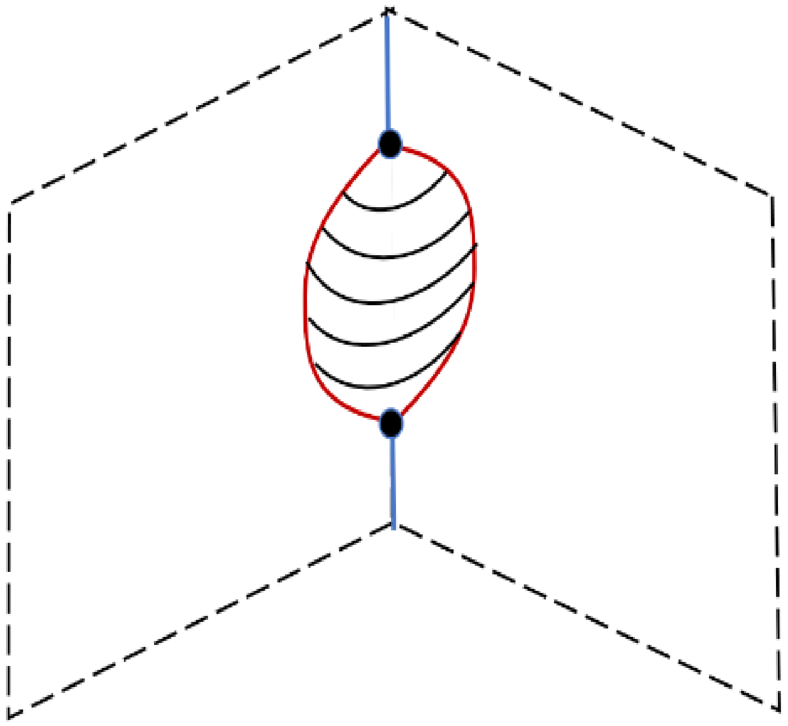
Comparison to Orbifold Method

We also compared the Susskind-Ulgum conical method to another proposal*, in which the angle deficit is produced by a Z_N orbifold which has opening angle $\beta = 2\pi/N$.
Since this is an on-shell solution, this gives *no* sphere contribution (even after continuing to $N \notin Z$).

The difference from the NLSM cone arises from the fact that twisted strings cannot pinch off @ tip.
This makes the noninteger N solution non-geometric, explaining nonunitary answer found by Witten 19.
The story might be different if twisted tachyons are allowed to condense before taking $N \to 1$,
as proposed by Dabholkar 02 based on Adams-Polchinski-Silverstein 01.

twisted strings in Z_3 orbifold:



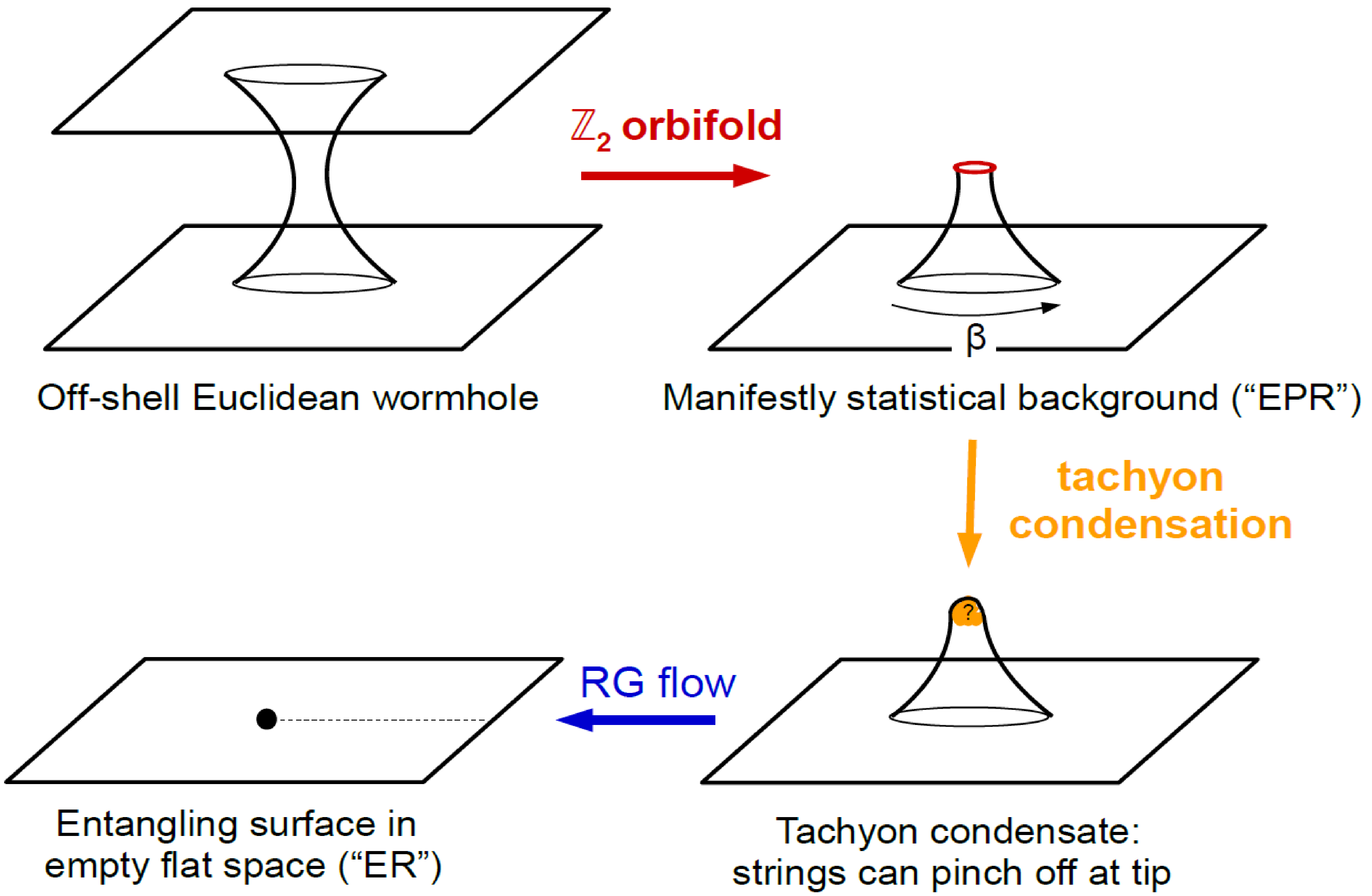


pinching-off process

(allowed for cone but not orbifold)

* Lowe-Strominger, Dabholkar, He-Numasawa-Takayanagi-Watanabe, Witten...

Possible ER = EPR construction?



Q & A

<u>Relation to string field theory (if asked):</u>

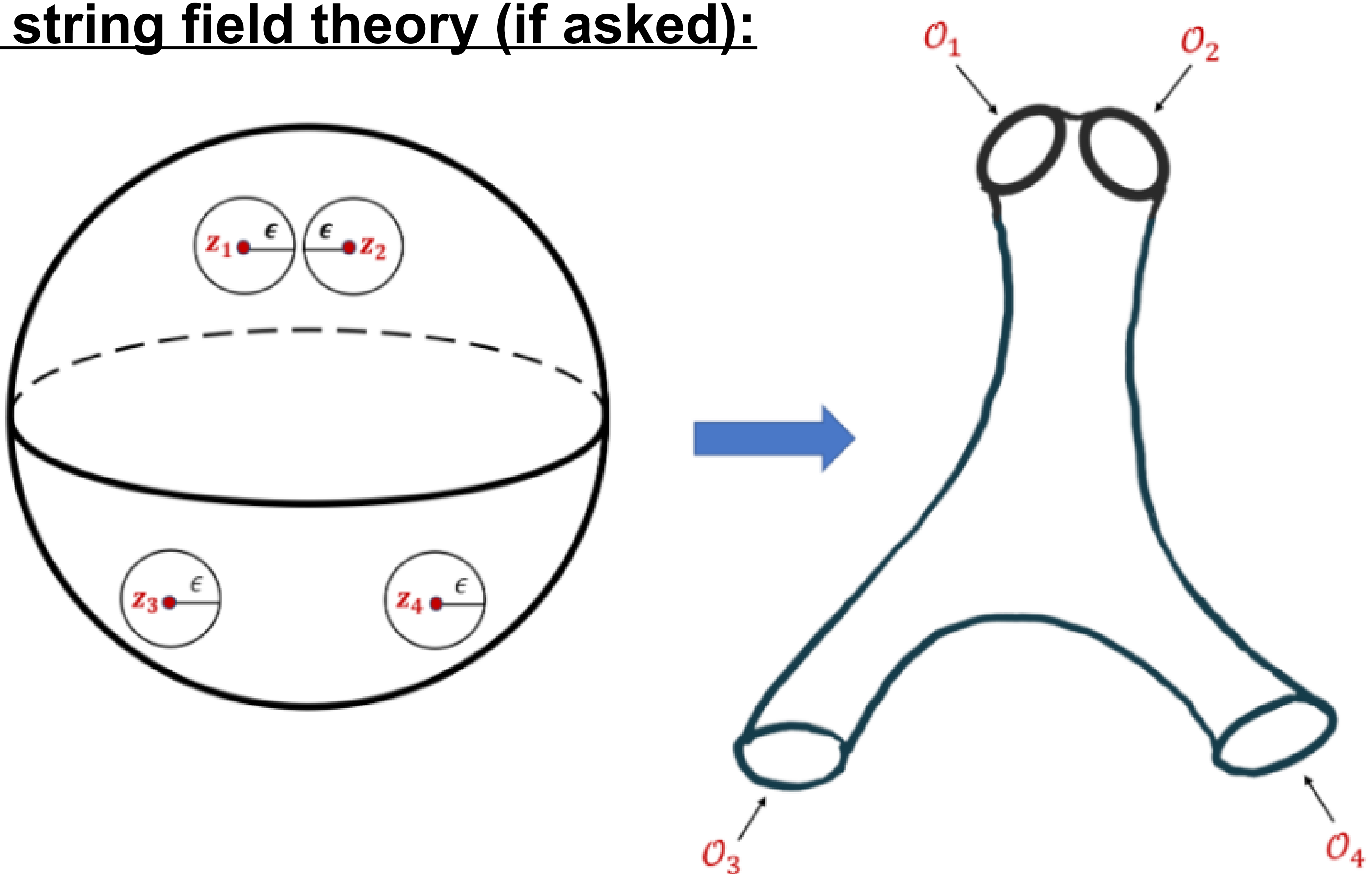


Figure 2: A conformal transformation which replaces the sphere regulated with a hard disk cutoff, with an open Riemann surface. The vertex operator insertions become state insertions on the boundaries. The length of the external and internal tubes is determined by the relative positioning of $z_1 \ldots z_4$ on the worldsheet, relative to our choice of Weyl frame ω. In fact this is the sole effect of ω and ϵ, as everything else is conformally invariant. In principle this conformal transformation allows one to re-express Tseytlin's off-shell formalism in the language of string field theory, but with an rather exotic rule for determining where to truncate the external propagators.