A fake explanation of sub-maximal chaos

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(chaos) (size) (symmetries) (traversable wormholes) (chord number) (q-deformations)

see also: 2208.07032 [HL], 1811.0258 [Berkooz, Isachenkov, Narovlansky, Torrents], 2108.04841 [Harlow & Wu], 1904.12820 [HL, Maldacena & Zhao]

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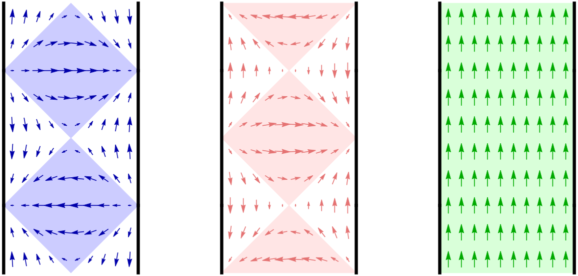
$\Rightarrow$ consequence of the near horizon symmetries.

For near-extremal black holes, $\mathfrak{sl}_2$ of the NAdS_2 throat:



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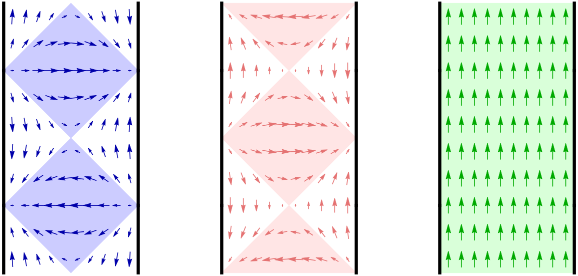
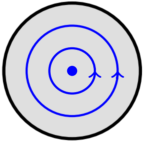
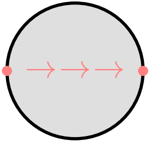
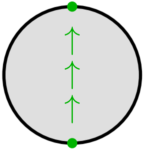
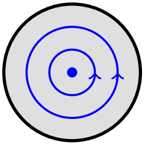
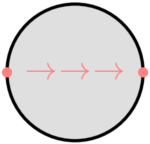
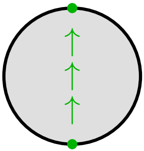


These generators move matter with respect to the Schwarzian boundary, preserving the length of the wormhole [HL, Maldacena, Zhao; Harlow & Wu].

$\mathsf{B} =$  $-\mathrm{i}\mathsf{P} =$  $\mathsf{E} =$ 

$$\mathrm{i}[\mathsf{B}, \mathsf{P}^{\pm}] = \mp \mathsf{P}^{\pm}, \quad \mathsf{P}^{\pm} = \mathsf{E} \pm \mathsf{P}, \quad \mathsf{B} \approx \frac{\beta}{2\pi}(H_R - H_L)$$

$\mathsf{B} =$  $-\mathrm{i}\mathsf{P} =$  $\mathsf{E} =$ 

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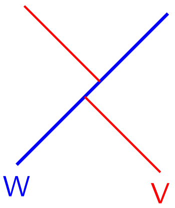
$$e^{\mathrm{i}(H_R - H_L)t}\mathsf{P}^{-}e^{-\mathrm{i}(H_R - H_L)t} = e^{\frac{2\pi}{\beta}t}\mathsf{P}^{-}$$

To understand why P^- is relevant, consider $2 \to 2$ gravitational scattering [Dray & 't Hooft; Shenker & Stanford; Gao-Jafferis-Wall; Lam *et al.*, . . .]:



Shapiro time delay $\Rightarrow$ $\mathrm{P}^{\pm}$ symmetry

In string theory, chaos is reduced [Shenker & Stanford]:

$$\text{chaos exponent} = \frac{2\pi v}{\beta}, \quad v = 1 - \# \left(\frac{\ell_s}{\ell_{\text{AdS}}} \right)^2 + \cdots$$

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Closely related to inelasticity.

Wish list: v in $\mathcal{N} = 4$ SYM at finite $(\lambda_{\text{'t Hooft}}, \beta)$.

A simpler model with a known chaos exponent is "large p SYK" [Maldacena & Stanford]:

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Goal: give a *bulk* interpretation of $v < 1$.

Strategy

1. scenic detour through double scaled SYK [Berkooz *et al.*, $\cdots$], where the "bulk Hilbert space" is known [HL].
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Fake geometry $\sim$ subtlety with the continuum limit.

SYK

N Majorana fermions, $p$articipating in the interaction

$$H = \mathrm{i}^{p/2} \sum_{1 \le i_1 < \cdots < i_p \le N} J_{i_1 \dots i_p} \psi_{i_1} \cdots \psi_{i_p}, \quad \left\langle J^2_{i_1 \dots i_p} \right\rangle \propto \mathcal{J}^2$$

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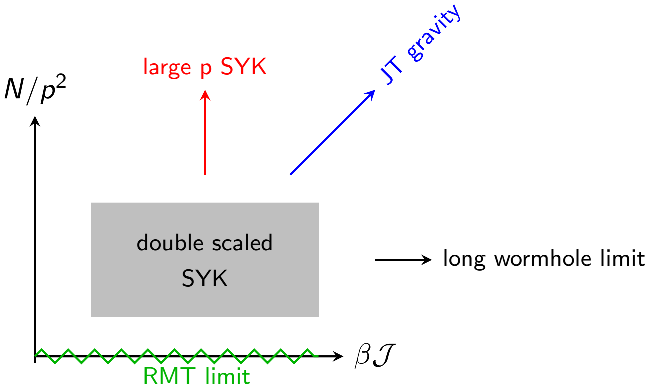
Take $N \to \infty, p \to \infty$. 2 remaining parameters: $(N/p^2, \beta\mathcal{J})$.

scaling limits of SYK



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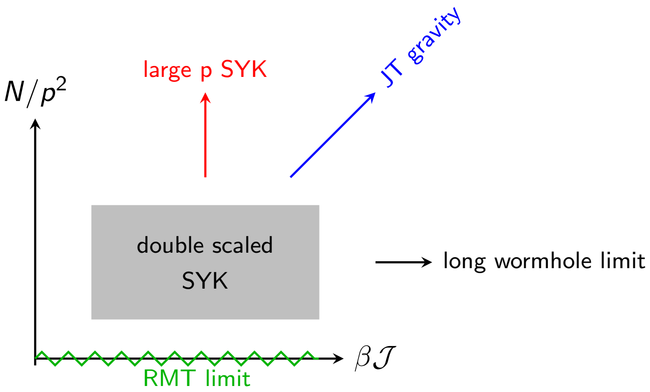

Convenient to introduce $\lambda = 2p^2/N$.

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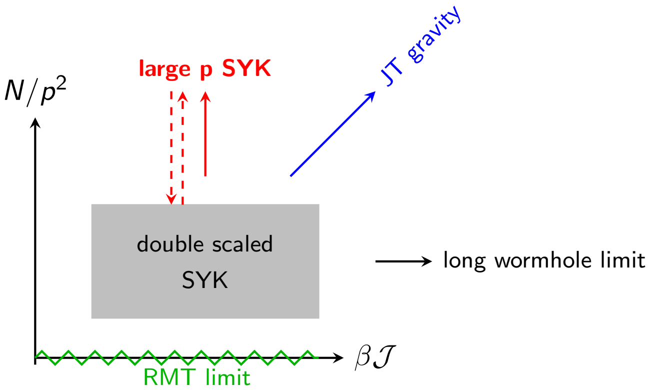
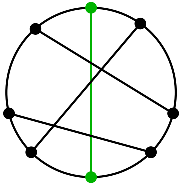


The advantage of the double scaled limit is that there is a new technique "chord diagrams" [Berkooz *et al.*, . . .] that can be used to solve the theory.

Chord rules

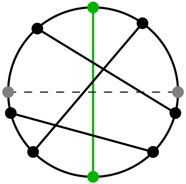
1. Draw Wick contractions[1] "chords" between like operators
2. Intersections between chords get a factor of $q = e^{-\lambda}$ (Hamiltonian) or $r = e^{-\lambda\Delta}$ (matter, $\Delta = p'/p$).
3. Sum over chord diagrams

$\mathrm{tr}\big(H^3 M H^3 M\big) \quad \supset$

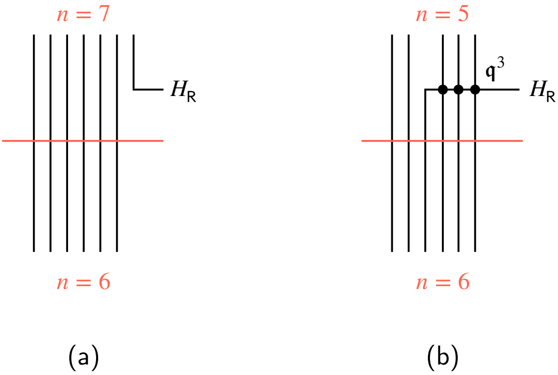


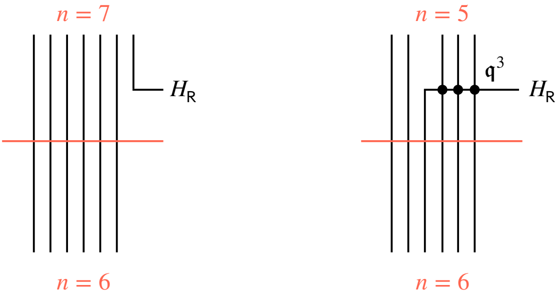
$= \quad q^2 r^3$

[1] Double intersections are forbidden



$$\Rightarrow \quad |\,|\wr|\,\rangle = |1, 1\rangle$$

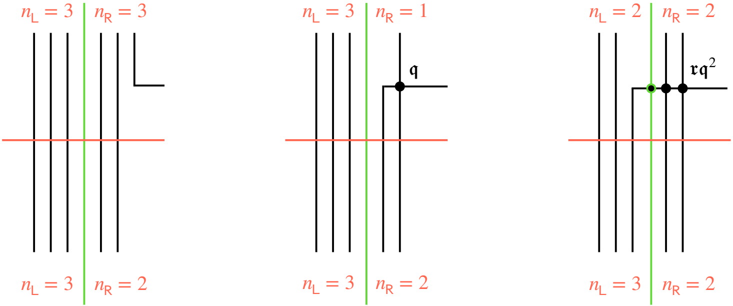

n = 7
H_R
n = 6
(a)
n = 5
q^3
H_R
n = 6
(b)



(a) (b)

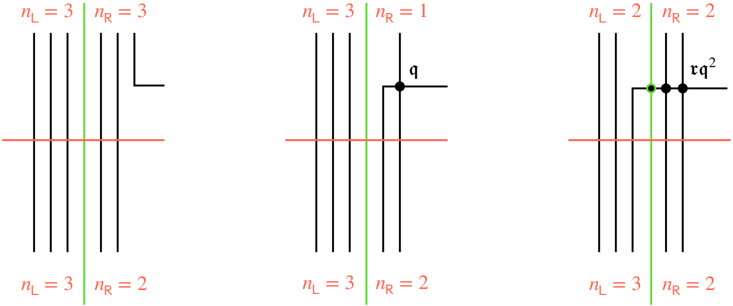
$$H = \mathfrak{a}^\dagger + \mathfrak{a}, \quad \mathfrak{a} = \alpha[n], \quad \alpha\,|n\rangle = |n-1\rangle$$

$$[n] = q^0 + q^1 + \cdots + q^{n-1} = \frac{1-q^n}{1-q}$$



(a) (b) (c)

$$H_R = \mathfrak{a}_R^\dagger + \mathfrak{a}_R, \quad \mathfrak{a}_R = \alpha_R[n_R] + \alpha_L q^\Delta q^{n_R}[n_L]$$



(a) (b) (c)

$$H_{\mathsf{R}} = \mathfrak{a}_{\mathsf{R}}^{\dagger} + \mathfrak{a}_{\mathsf{R}}, \quad \mathfrak{a}_{\mathsf{R}} = \alpha_{\mathsf{R}}[n_{\mathsf{R}}] + \alpha_{\mathsf{L}} q^{\Delta} q^{n_{\mathsf{R}}}[n_{\mathsf{L}}]$$

$$H_{\mathsf{L}} = \mathfrak{a}_{\mathsf{L}}^{\dagger} + \mathfrak{a}_{\mathsf{L}}, \quad \mathfrak{a}_{\mathsf{L}} = \alpha_{\mathsf{L}}[n_{\mathsf{L}}] + \alpha_{\mathsf{R}} q^{\Delta} q^{n_{\mathsf{L}}}[n_{\mathsf{R}}]$$

The chord algebra

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$$\{H_L, H_R, \bar{n}\}$$

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$\bar{n}$ counts the number of chords (weighted by Δ). Discrete analog of the length of the Einstein-Rosen bridge.

Microscopic interpretation of chord number

Operator size [Roberts-Stanford-Streicher, Qi-Streicher] is measured by the 2-sided operator:

$$\text{size} = \frac{1}{2} \sum_{\alpha=1}^{N} \left(1 + i\psi_\alpha^{L} \psi_\alpha^{R} \right).$$

Let $\bar{n}$ be the total chord number, weighted by dimension:

$$\boxed{\bar{n} = \text{size}/p}$$

Each H chord is associated with an operator of size p.

q-deformed commutator:

$$[A, B]_q \equiv AB - qBA$$

Writing $H_L = \mathfrak{a}_L^\dagger + \mathfrak{a}_L$ and $H_R = \mathfrak{a}_R^\dagger + \mathfrak{a}_R$,

$$[\mathfrak{a}_L, \mathfrak{a}_R] = [\mathfrak{a}_L^\dagger, \mathfrak{a}_R^\dagger] = 0, \tag{1}$$

$$[\bar{n}, \mathfrak{a}_{\mathrm{L/R}}^\dagger] = \mathfrak{a}_{\mathrm{L/R}}^\dagger, \quad [\bar{n}, \mathfrak{a}_{\mathrm{L/R}}] = -\mathfrak{a}_{\mathrm{L/R}} \tag{2}$$

$$[\mathfrak{a}_L, \mathfrak{a}_R^\dagger] = [\mathfrak{a}_R, \mathfrak{a}_L^\dagger] = q^{\bar{n}} \tag{3}$$

$$[\mathfrak{a}_{\mathrm{L/R}}, \mathfrak{a}_{\mathrm{L/R}}^\dagger]_q = 1. \tag{4}$$

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$$D(\mathfrak{a}_L^\dagger) = \mathfrak{a}_L^\dagger \otimes 1, \tag{5}$$

$$D(\mathfrak{a}_L) = \mathfrak{a}_L \otimes 1 + q^\Delta q^{\bar{n}} \otimes \mathfrak{a}_L, \tag{6}$$

$$D(\bar{n}) = \bar{n} \otimes 1 + 1 \otimes \bar{n} + \Delta(1 \otimes 1) \tag{7}$$

This coproduct has a simple physical interpretation:

$$\delta : \left|\mathbf{I}{\color{red}\wr}\mathbf{II}\right\rangle \otimes \left|\mathbf{I}{\color{red}\wr}\mathbf{I}\right\rangle \to \left|\mathbf{I}{\color{red}\wr}\mathbf{II}{\color{blue}\wr}\mathbf{I}{\color{red}\wr}\mathbf{I}\right\rangle$$

δ joins two wormholes; δ^{-1} splits:

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Coproduct:

$$D(a) = \delta^{-1} \cdot a \cdot \delta$$

Related to the factorisation problem [..., Harlow and Jafferis, ...].

Representations of $\mathcal{A}_{\mathsf{chord}}$

What are the chord algebra irreps?

1. Short irrep, consisting of the "empty wormhole" states:

$$|n\rangle\,, \quad n \in \mathbb{Z}_{\geq 0}$$

2. 1-particle irrep (highest weight irrep $\mathfrak{a}_L |\wr\rangle = \mathfrak{a}_R |\wr\rangle = 0.$):

$$|n_L, n_R\rangle = |\mathbf{||}\cdots\mathbf{|}\wr\mathbf{||}\cdots\mathbf{|}\rangle$$

The Hilbert space decomposes into a sum over these irreps. What about multi-particle states?

Coproduct builds multi-particle reps of the chord algebra.

We solved the Clebsch-Gordan problem of decomposing multi-particle reps into irreps:

$$|\text{multi-chord state}\rangle = \sum \gamma_r \, |\text{irreps}\rangle \tag{8}$$

We define 4-pt "chord blocks" $=$ projector onto a particular irrep.
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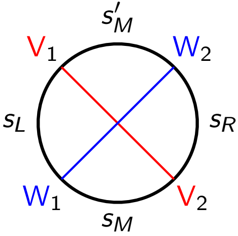
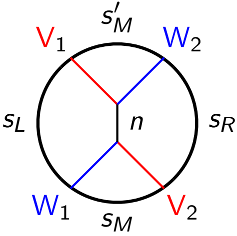
These blocks enjoy crossing symmetry, thanks to co-associativity of the co-product:

$$(D \otimes \mathbf{1}) \cdot D = (\mathbf{1} \otimes D) \cdot D$$

[Gives the Hilbert space interpretation of expressions in Jafferis, Kolchmeyer, Mukhametzhanov & Sonner]



$$= \sum_{n=0}^{\infty} \tilde{\gamma}_n$$



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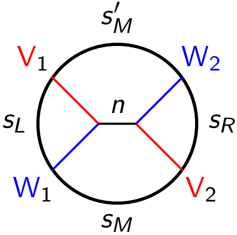

The intermediate operators $[VW]_n$ are analogous to double trace primaries, schematically $\sim VH^nW$. $\Delta_n = \Delta_V + \Delta_W + n$.

Two kinds of double traces $[VW]_n$ and $[WV]_n$. Twice the number we would expect from bulk free field theory in $AdS_2 \Rightarrow$ inelastic states

Fake geometry from algebra

Let's come back to the symmetries of large p SYK at finite temp.

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Look for a subalgebra $\subset \mathcal{A}_{\text{chord}}$ that commutes with $\bar{n}$:

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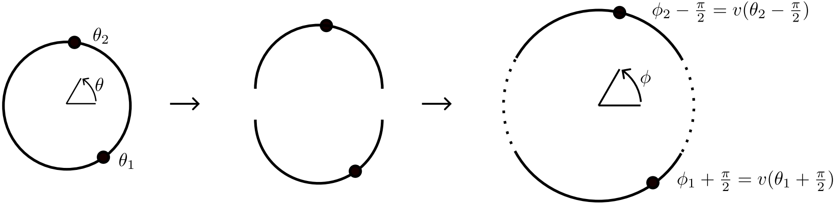
Define $c = q^{\bar{n}/2}$. Take $\lambda \to 0$, find an $\mathfrak{sl}_2$:

$$\mathsf{E} \doteq -\tfrac{1}{2c}\left(J_{LL} + J_{RR}\right), \quad \mathsf{B} \doteq \tfrac{1}{2c\sqrt{1-c^2}}\left(J_{LL} - J_{RR}\right)$$
$$\mathsf{P} \doteq \tfrac{\mathrm{i}}{2\sqrt{1-c^2}}\left(J_{LR} - J_{RL}\right)$$

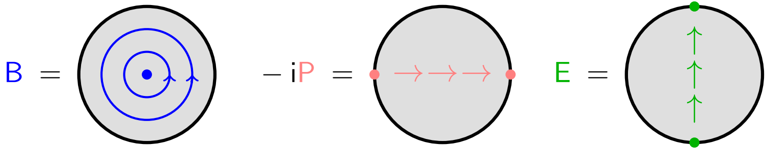
Valid at all temperatures

We studied how these generators act on boundary operators. What we found surprised us.



$$\langle \mathcal{O}(\theta_2) \mathsf{B} \mathcal{O}(\theta_1) \rangle = \partial_{\phi_1} \langle \mathcal{O}(\theta_2) \mathcal{O}(\theta_1) \rangle$$
$$\langle \mathcal{O}(\theta_2) \mathsf{E} \mathcal{O}(\theta_1) \rangle = (\cos(\phi_1) \partial_{\phi_1} - \Delta \sin \phi_1) \langle \mathcal{O}(\theta_2) \mathcal{O}(\theta_1) \rangle$$
$$\langle \mathcal{O}(\theta_2) \mathsf{P} \mathcal{O}(\theta_1) \rangle = \mathrm{i}(\sin(\phi_1) \partial_{\phi_1} + \Delta \cos \phi_1) \langle \mathcal{O}(\theta_2) \mathcal{O}(\theta_1) \rangle$$

The circumference of the fake disk is precisely $\beta_{\mathsf{fake}} = \beta / v$.



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$$e^{\mathrm{i}(H_R - H_L)t}\mathrm{P}^{-}e^{-\mathrm{i}(H_R - H_L)t} = e^{\frac{2\pi v}{\beta}t}\mathrm{P}^{-}$$

There is a “geometric” explanation of the sub-maximal exponent but it is unfamiliar.

Ward identites: *finite temp* 2-pt function is conformally covariant on the fake circle:

$$\langle \mathcal{O}(\theta_2)\mathcal{O}(\theta_1)\rangle \propto \frac{1}{\sin^{2\Delta}(\frac{\phi_2-\phi_1}{2})} \quad \checkmark$$

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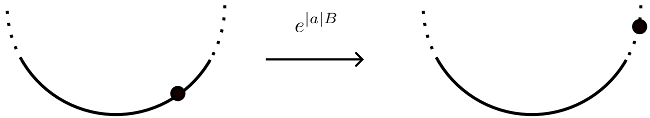
4-pt function ⇒ commutator OTOC is only a function of the *fake* cross ratio.

$$\frac{\langle [W_2, V_4][V_3, W_1]\rangle}{\langle W_2 W_1\rangle\langle V_4 V_3\rangle} = 2\lambda\Delta_W\Delta_V\frac{1+\chi}{1-\chi} = \sum_k c_k^2 \mathcal{F}_{\Delta_V,\Delta_W,k}(\chi),$$

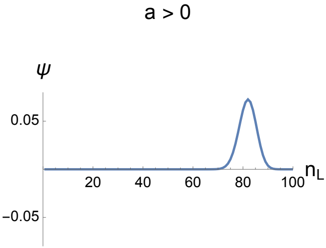
$$\chi = \frac{\sin\frac{\phi_{13}}{2}\sin\frac{\phi_{42}}{2}}{\sin\frac{\phi_{14}}{2}\sin\frac{\phi_{32}}{2}}$$

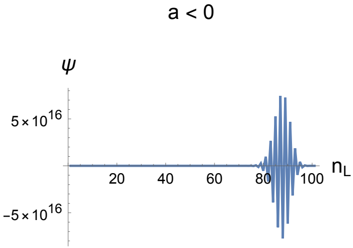
c_k^2 ⇒ Clebsch-Gordan problem described earlier.

If ϕ starts out in the physical region, can act with a symmetry generator to leave the physical region:



What do the states look like in the chord Hilbert space?





Note that the y-axis rescaled by 10^{18} between the two figures.

Two pictures:

We checked (numerically and analytically) that such states reproduce the fake disk predictions as $\lambda \to 0$.

Fake disk: makes sub-max chaos manifest

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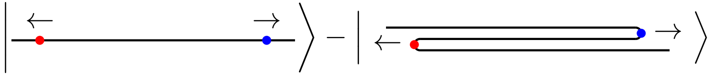

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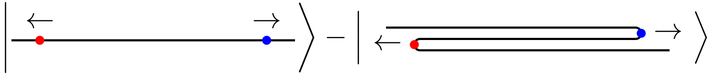
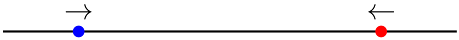


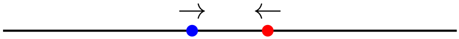
Question: general lessons for sub-max chaos?

backup slides



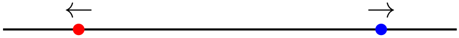
(9)

As time progresses forwards, the two particles approach each other, the middle distance ℓ_M shrinks:



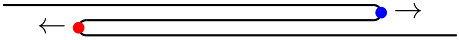
(10)

and the two (naively) pass through each other:



(11)

More accurate picture:



(12)

fakeness in $p = 4$ SYK

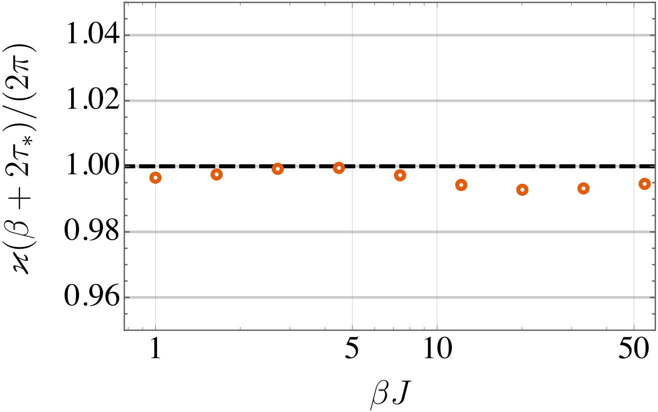


figure 9b of Gu, Kitaev, & Zhang