Effective Field Theory Methods for Gravity #2

John Donoghue
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TRISEP

Review: Constructing GR as QFT

Global symmetries and currents
Using current as source

$\psi \to U(x)\psi$ local symmetry

$D_\mu \psi \to U(x) D_\mu \psi$ covariant deriv.

$$[D_\mu, D_\nu] = ig \underline{\underline{F}}_{\mu\nu} = ig\left[\partial_\mu \underline{\underline{A}}_\nu - \partial_\nu \underline{\underline{A}}_\mu + g[\underline{\underline{A}}_\mu, \underline{\underline{A}}_\nu]\right]$$

$T_{\mu\nu}$ as source $\Rightarrow$ local coord changes

$$dx'^\mu = J^\mu{}_\nu(x) dx^\nu$$

$$g'_{\mu\nu} = J^{-1}{}_\mu{}^\alpha(x) J^{-1}{}_\nu{}^\beta(x) g_{\alpha\beta}$$

If $V'^\mu = J^\mu{}_\nu(x) V^\nu$ want $D_\mu V'^\lambda = J_\mu^{-1\nu} J^\lambda{}_\sigma (D_\nu V^\sigma)$

or $D_\mu V^\lambda = \partial_\mu V^\lambda + \Gamma_{\mu\nu}{}^\lambda V^\nu$ ← gauge trans

with $\Gamma'_{\mu\nu}{}^{\lambda} = (J^{-1})^{\mu'}_{\mu}(J^{-1})^{\nu'}_{\nu}J^{\lambda}_{\lambda'}\left(\Gamma_{\mu'\nu'}{}^{\lambda'} + (J^{-1})^{\lambda'}_{\sigma}\partial_{\mu'}J^{\sigma}_{\nu'}\right)$

Field strength

$$[D_\mu, D_\nu] V^\alpha = R_{\mu\nu\beta}{}^\alpha V^\beta$$

$$R_{\mu\nu\alpha}{}^\beta = \partial_\mu \Gamma_{\nu\alpha}{}^\beta - \partial_\nu \Gamma_{\mu\alpha}{}^\beta + \Gamma_{\mu\rho}{}^\beta \Gamma_{\nu\alpha}{}^\rho - \Gamma_{\nu\rho}{}^\beta \Gamma_{\mu\alpha}{}^\rho$$

$$\left.\begin{array}{l}\text{If metricity } D_\mu g_{\alpha\beta} = 0 \\ + \text{ symmetry } \Gamma_{\mu\nu}{}^\lambda = \Gamma_{\nu\mu}{}^\lambda\end{array}\right\} \underline{\text{obtain GR}}$$

with

$$\Gamma_{\mu\nu}{}^\lambda = \frac{1}{2} g^{\lambda\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right]$$

Cousins of GR

- No symmetry $\Gamma_{\mu\nu}^{\ \ \lambda} \neq \Gamma_{\nu\mu}^{\ \ \lambda}$ — torsion "Einstein Cartan"
- Non metricity $D_\mu g_{\alpha\beta} \neq 0$ "metric affine"

With $$S_g = \int d^4x \sqrt{-g}\, \frac{2}{\kappa^2} R + S_m$$

$$\kappa^2 = 32\pi G$$

obtain $G_{\mu\nu} = 8\pi G\, T_{\mu\nu}$ classical

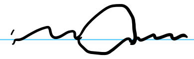
$$\left.\begin{aligned} i\partial_t \psi &= \left[-\frac{\nabla^2}{2m} + m\phi_g\right]\psi \\ \dot{\vec{p}} &= -i[H, \vec{p}] = -m\vec{\nabla}\phi_g = m\vec{a} \end{aligned}\right\} \text{QM}$$

g $\Rightarrow$ $V(r) = -\dfrac{G m_1 m_2}{r}$ QFT

and Feynman rules

Still to be done — general background $\bar{g}_{\mu\nu}$
— ghosts

First calculation:



$$\sim \quad \Delta\mathcal{L} = \frac{1}{\epsilon}\left[a R^2 + b R_{\mu\nu} R^{\mu\nu}\right]$$

$\leftarrow (\partial g)^4$

$\neq \# R$

EFT via two examples

1) QED with heavy top quark

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi$$

$$Z = \int [dA_\mu][d\psi][d\bar{\psi}]\, e^{i\int d^4x\, \mathcal{L}(A,\psi)}$$

$$= \int [dA_\mu]_{\Lambda = m_t}\, e^{i\int d^4x\, \mathcal{L}_{eff}(A)}$$

with $\mathcal{L}_{eff} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{(\overset{Q_t}{\frac{2}{3}} e)^2}{240\pi^2 m_t^2} F_{\mu\nu} \Box F^{\mu\nu} + \ldots$

2) Linear sigma model

$$\mathcal{L}_\sigma = \frac{1}{2}\partial_\mu\sigma\,\partial^\mu\sigma + \frac{1}{2}\partial_\mu\vec{\phi}\,\partial^\mu\vec{\phi} + \frac{\mu^2}{2}(\sigma^2 + \vec{\phi}^2) - \frac{\lambda}{4}(\sigma^2 + \vec{\phi}^2)^2$$

with SSB $\langle\sigma\rangle = v = \sqrt{\frac{\mu^2}{\lambda}}$

$$Z = \int [d\sigma][d\vec{\phi}]\, e^{i\int d^4x\, \mathcal{L}_\sigma(\sigma,\vec{\phi})}$$

$$= \int [d\vec{\pi}]_{\Lambda = m_\sigma}\, e^{i\int d^4x\, \mathcal{L}_{eff}(\vec{\pi})}$$

With $\mathcal{L} = \frac{v^2}{4} \mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U) + \ell_1 [\mathrm{Tr}(\partial_\mu U^\dagger \partial^\mu U)]^2 + \ell_2 \mathrm{Tr}[\partial_\mu U^\dagger \partial_\nu U]\, \mathrm{Tr}(\partial^\mu U^\dagger \partial^\nu U)$

with

$U = \exp\left[\frac{i\vec{\tau}\cdot\vec{\pi}}{v}\right]$ and

$$\ell_1^r = \frac{v^2}{8m_\sigma^2} + \frac{1}{192\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{35}{6}\right]$$

$$\ell_2^r = \frac{1}{384\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{11}{6}\right].$$

!

QED c heavy top

$$\Pi(q) = \frac{e_0^2}{12\pi^2}\left[\frac{1}{\epsilon} + \ln(4\pi) - \gamma - 6\int_0^1 dx\, x(1-x)\ln\left(\frac{m^2 - q^2 x(1-x)}{\mu^2}\right) + \mathcal{O}(\epsilon)\right]$$

$$= \frac{e_0^2}{12\pi^2}\begin{cases} \frac{1}{\epsilon} + \ln(4\pi) - \gamma + \frac{5}{3} - \ln\frac{-q^2}{\mu^2} + \cdots & (|q^2| \gg m^2), \\ \frac{1}{\epsilon} + \ln(4\pi) - \gamma - \ln\frac{m^2}{\mu^2} + \frac{q^2}{5m^2} + \cdots & (m^2 \gg |q^2|). \end{cases}$$

$$\frac{e_0^2}{q^2[1+\Pi(q^2)]}$$

$\ln m_t^2$

Renormalize $\frac{1}{\epsilon}$ disappear

Look like Σ depends on m_t ?

Renormalize $\Rightarrow$ measure it

$$\frac{e_0^2}{4\pi[1+\Pi(0)]} = \frac{e^2}{4\pi} = \frac{1}{137}$$

$\Rightarrow$ no $\ln m_t$

residual $\frac{q^2}{m_t^2}$

Appelquist Carazzone thm

Heavy effect renormalize parameter

or suppressed $1/m^2$

EFT Logic – Uncertainty princ.

Heavy $\longrightarrow$ Local

$\Longrightarrow$ Local $\mathcal{L}$

$\Longrightarrow$ coupling consts.

Match residual effect to Local $\mathcal{L}$

$$\frac{g^2}{M_t^2}$$

$$F_{\mu\nu} \frac{\Box}{m_t^2} F^{\mu\nu}$$

Using

Power correction $\displaystyle\int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x} \left[q^2\right]^n$ ✓

$$= \Box^n \, \delta^4(x)$$

Non analytic $\displaystyle\int \frac{d^4q}{(2\pi)^4} e^{iq\cdot x} \ln(q^2)$

$$= L(x) \sim \frac{1}{x^4}$$

2) Nonlinear σ model

a) Usual notation $\quad \sigma = v + \tilde{\sigma}$

$$\mathcal{L} = \underbrace{\frac{1}{2}\partial_\mu\vec{\phi}\,\partial^\mu\vec{\phi}}_{GB} + \frac{1}{2}\left[(\partial_\mu\tilde{\sigma})^2 - m_\sigma^2\tilde{\sigma}^2\right] - \lambda v\tilde{\sigma}(\tilde{\sigma}^2+\vec{\phi}^2) - \frac{\lambda}{4}(\tilde{\sigma}^2+\vec{\phi}^2)^2$$

b) Better notation

$$\Sigma = \sigma + i\vec{\tau}\cdot\vec{\phi} \quad \leftarrow \text{Pauli}$$

$$\frac{1}{2}\mathrm{Tr}(\Sigma^\dagger\Sigma) = \sigma^2 + \vec{\phi}^2$$

$$\mathcal{L} = \frac{1}{4}\mathrm{Tr}\left(\partial_\mu\Sigma^\dagger\partial^\mu\Sigma\right) + \frac{\mu^2}{4}\mathrm{Tr}(\Sigma^\dagger\Sigma) - \frac{\lambda}{16}(\mathrm{Tr}\,\Sigma^\dagger\Sigma)^2$$

Shows $\underbrace{\Sigma \to L^\dagger\Sigma R}_{SU(2)}$ $\qquad \mathrm{Tr}(\Sigma^\dagger\Sigma)_{min}$

c) Best notation

$$\Sigma = (v+S)\,U \qquad U = e^{\frac{i\vec{\tau}\cdot\vec{\pi}}{v}}$$

$$\begin{cases} S = \tilde{\sigma} + (\ldots) \\ \vec{\pi} = \vec{\phi} + (\quad) \end{cases} \qquad \text{KE same}$$

$$\mathcal{L} = \frac{(v+S)^2}{4}\underline{\mathrm{Tr}(\partial_\mu U\,\partial^\mu U^\dagger)} - \lambda v S^3 - \lambda S^4 + \frac{1}{2}\left[(\partial S)^2 - m_\sigma^2 S^2\right]$$

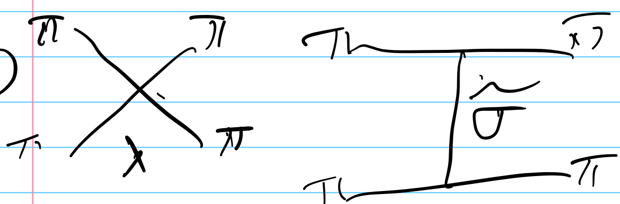
no approx.

Haag's thm Names don't matter

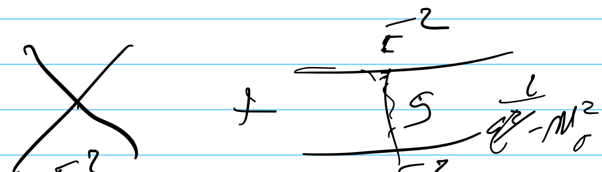
Scattering amps identical on shell

K.F. unchanged

(a)+(c) give same amps PSM Ch 4

a) 

$$= 1 - 1 + \frac{g^2}{m_\sigma^2} + \frac{g^4}{m_\sigma^4}$$

c) 

$$\frac{g^2}{m_\sigma^2} \left(+ \frac{g^4}{m_\sigma^4} \right)$$

$\frac{E^2}{v^2}$ $\frac{E^4}{v^4}$

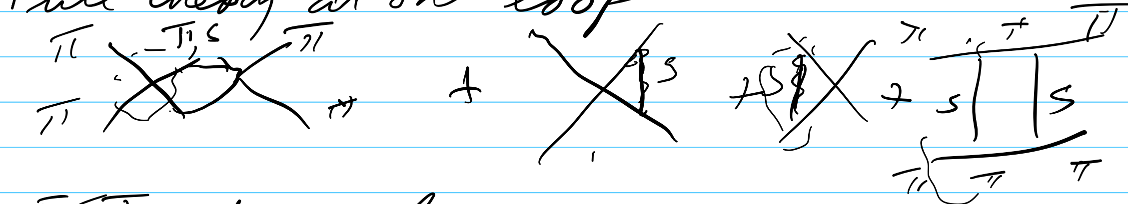
$$\mathcal{L} = \frac{v^2}{4} \mathrm{Tr}\, \partial_\mu U \partial^\mu U^\dagger + \frac{v^2}{8 m_\sigma^2} \left[\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) \right]^2 + \cdots$$

But EFT is full QFT

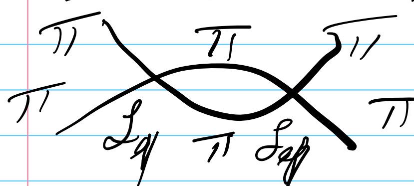
$$Z = \int [ds][d\pi]\, e^{i\mathcal{L}(s,\pi)}$$

$$= \int d\pi\, e^{i\mathcal{L}_{eff}(\pi)}$$ light DOF

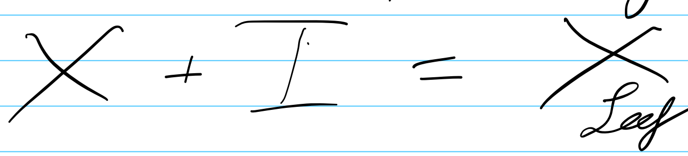
Full theory at one loop



EFT at one loop



Equiv at low loop energies *



High loop energies differ

- local $\mathcal{L}$

coeff

EFT at work

Most general $\mathcal{L}_{eff}$

$$\mathcal{L} = \frac{v^2}{4} \mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger) + \ell_1 \left[\mathrm{Tr}(\partial_\mu U \partial^\mu U^\dagger)\right]^2$$

$$+ \ell_2 \, \mathrm{Tr}(\partial_\mu U \partial_\nu U^\dagger) \mathrm{Tr}(\partial^\mu U \partial^\nu U^\dagger)$$

$$+ \ldots$$

The full theory, at low energy

$$\mathcal{M}_{\text{full}} = \frac{t}{v^2} + \left[\frac{1}{m_\sigma^2 v^2} - \frac{11}{96\pi^2 v^4}\right] t^2 - \frac{1}{144\pi^2 v^4}[s(s-u) + u(u-s)] - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{m_\sigma^2} + s(s-u)\ln\frac{-s}{m_\sigma^2} + u(u-s)\ln\frac{-u}{m_\sigma^2}\right].$$

$t = (p_1 - p_3)^2$

$s = (p_1 + p_2)^2$

$u = -s - t$

The EFT has diff High E divergences, but can be renormalized

$$\mathcal{M}_{\text{eff}} = \frac{t}{v^2} + \left[8\ell_1^r + 2\ell_2^r + \frac{5}{192\pi^2}\right]\frac{t^2}{v^4} + \left[2\ell_2^r + \frac{7}{576\pi^2}\right][s(s-u) + u(u-s)]/v^4 - \frac{1}{96\pi^2 v^4}\left[3t^2 \ln\frac{-t}{\mu^2} + s(s-u)\ln\frac{-s}{\mu^2} + u(u-s)\ln\frac{-u}{\mu^2}\right]$$

with identification

$$\ell_1^r = \ell_1 + \frac{1}{384\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right]$$

$$\ell_2^r = \ell_2 + \frac{1}{192\pi^2}\left[\frac{2}{4-d} - \gamma + \ln 4\pi\right]$$

} renormalization

The theories <u>match</u> exactly if

$$\ell_1^r = \frac{v^2}{8m_\sigma^2} + \frac{1}{192\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{35}{6}\right]$$

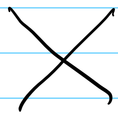
$$\ell_2^r = \frac{1}{384\pi^2}\left[\ln\frac{m_\sigma^2}{\mu^2} - \frac{11}{6}\right].$$

} ✓

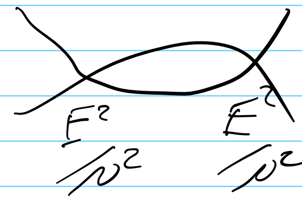
Match on measure

Power counting theorem

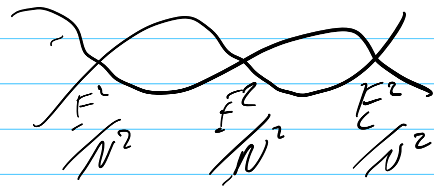
— Weinberg



$\sim \frac{E^2}{\Lambda^2}$



$\sim \frac{E(E)}{\Lambda^4} \sim \frac{E^4}{\Lambda^4}$



$\sim \frac{E^6}{\Lambda^6}$

Background Field Method

Game plan:

$$U(x) = \bar{U}(x)\, e^{i\Delta(x)} \qquad \Delta = \tau^a \Delta^a$$

$\Delta(x)$: quantum field

$$\mathcal{L} = \mathcal{L}(\bar{U}) + \underbrace{\mathcal{L}_1(\bar{U},\Delta)}_{\text{linear in }\Delta} + \underbrace{\mathcal{L}_2(\bar{U},\Delta)}_{\text{2nd order}} + \cdots$$

linear in Δ vanish by eq of motion

$$\mathcal{L}_2 = \frac{1}{2}\Delta^a \left[d_\mu d^\mu + \sigma(x) \right]_{ab} \Delta^b$$

d_μ ← $\bar{U}$, $\sigma(x)$ ← $\bar{U}$

$$d_\mu = \partial_\mu + \Gamma_\mu$$

Pull out divergences

$$\mathcal{L}_{div} = \frac{1}{16\pi^2}\frac{\mathrm{Tr}}{\varepsilon}\left(\left[\frac{1}{12}\Gamma_{\mu\nu}\Gamma^{\mu\nu} + \frac{1}{2}\sigma^2 \right] \right)$$

← "a_2 coeff"

$$\Gamma_{\mu\nu} = [d_\mu, d_\nu]$$

This local $\mathcal{L} \Rightarrow \underline{l_1} [\mathrm{Tr}(\;)]^2 + \underline{l_2}\, \mathrm{Tr}\,\mathrm{Tr}$

Renormalize <u>all</u> processes at once

$$d^{ab}_\mu = \delta^{ab}\partial_\mu + \Gamma^{ab}_\mu,$$
$$\Gamma^{ab}_\mu = -\frac{1}{4}\,\mathrm{Tr}\left([\lambda^a,\lambda^b]\left(\bar U^\dagger\partial_\mu \bar U + i\bar U^\dagger \ell_\mu \bar U + i r_\mu\right)\right),$$
$$\sigma^{ab} = \frac{1}{8}\,\mathrm{Tr}\left(\{\lambda^a,\lambda^b\}\left(\chi^\dagger \bar U + \bar U^\dagger\chi\right) + [\lambda^a, \bar U^\dagger D_\mu \bar U][\lambda^b, \bar U^\dagger D^\mu \bar U]\right).$$

$\nwarrow r^a_\mu$

Evaluate divergence:

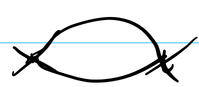
1) $\int [d\Delta]\, e^{i\int d^4x\, \Delta[dd+\sigma]\Delta}$

$$= \left(\det[d_\mu d^\mu + \sigma]\right)^{-1/2} = e^{-\frac{1}{2}\mathrm{Tr}\ln[d^2+\sigma]}$$

$$\mathrm{Tr}\ln[d_\mu d^\mu + \sigma] = \mathrm{Tr}\ln(\Box + \sigma)$$
$$= \mathrm{Tr}\ln \Box\left(1 + \frac{1}{\Box}\sigma\right)$$
$$= \mathrm{Tr}\ln\Box + \mathrm{Tr}\frac{1}{\Box}\sigma + \mathrm{Tr}\left(\frac{1}{\Box}\sigma\frac{1}{\Box}\sigma\right) + \ldots$$

$$\langle x|\frac{1}{\Box}|y\rangle = D_F(x-y)$$


$\sim \int d^4x\, D(x-x)\sigma(x)$

$\left(\frac{1}{\Box}\sigma\frac{1}{\Box}\sigma\right) \sim$ 

$\int d^4x\, d^4y\, \sigma(x)\, D(x-y)\, \sigma(y)\, D(y-x)$

$\Longrightarrow$ Feynman diagrams $\Rightarrow a_2$

Or

Heat kernel

$\mathcal{D}$

$$H(x,\tau) = \langle x | e^{-\mathcal{D}\tau} | x \rangle$$

$$\mathcal{D} = d_\mu d^\mu + \sigma + m^2$$

$$= \frac{i}{(4\pi)^{d/2}} \frac{e^{-\tau m^2}}{\tau^{d/2}} \Big[\underbrace{a_0 + a_1 \tau + a_2 \frac{\tau^2}{2}}_{\text{Seeley DeWitt}} + \cdots \Big]$$

$$\langle x | \ln \mathcal{D} | x \rangle = -\int \frac{d\tau}{\tau} \langle x | e^{-\tau \mathcal{D}} | x \rangle$$

$$= \frac{-i}{(4\pi)^{d/2}} \sum m^{d-2n} \Gamma(n - \tfrac{d}{2}) a_n(x)$$

$$\sim \Gamma(2 - \tfrac{d}{2}) a_2$$

$\Longrightarrow a_2(\quad)$