EFT Methods for General Relativity

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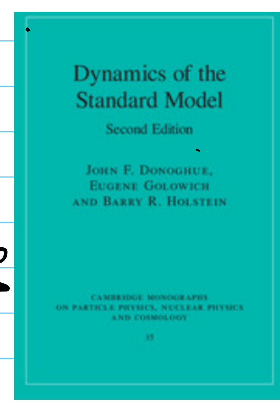
Roughly 3 blocks:

1) GR as a QFT
   - to think like a field theorist
   - Feynman rules
2) Effective Field Theory
   - as a full QFT

3) GR as an EFT
   - tie together
   - limits

Along the way: Background Field Method
Path Integrals
Maybe Renormalization Theory
Heat Kernel Techniques

Web site: (Donoghue UMass or Donoghue GRQFT)
   - References (EPFL lecture; Dynamics of SM ...)
   - Notes, summaries, link
   - also note ISQG lectures (~10 hours)
   - plus EFT lectures at PI (~6 hrs)



Note: "Dynamics of Standard Model" is free
   - JFD, Golowich + Holstein
   - link is on webpage

Constructing theories

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi$$

Symmetry $\Psi \to e^{i\theta}\Psi$

Current $e\bar{\Psi}\gamma_\mu\Psi = eJ_\mu$

Gauge symmetry $\Psi \to e^{i\theta(x)}\Psi$

New field $D_\mu\Psi \to e^{i\theta(x)} D_\mu\Psi$

$$\mathcal{L} = \bar{\Psi}(iD\!\!\!\!/ - m)\Psi \to \mathcal{L}$$

$$D_\mu = \partial_\mu + ieA_\mu$$

$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu\theta$$

Now $\mathcal{L} = \bar{\Psi} \ldots\ldots eA\!\!\!/\,\Psi$

$$\frac{\partial\mathcal{L}}{\partial A_\mu} \sim \underbrace{\ldots e\bar{\Psi}\gamma_\mu\Psi}_{\text{source}} \quad \checkmark$$

Use

$$[D_\mu, D_\nu]\Psi = ie\underbrace{F_{\mu\nu}}_{\text{invariant}}\Psi$$

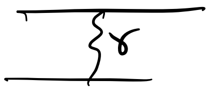
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\Psi}(iD\!\!\!\!/ - m)\Psi$$

$$\Longrightarrow \underbrace{\partial^\mu F_{\mu\nu}} = eJ_\mu$$

$\cancel{O_{\mu\nu}\,\partial^\nu\,{}_\lambda = \delta_\mu}$

$$O_{\mu\nu}A^\nu = \underbrace{\Box A_\nu - \partial_\nu(\partial_\mu A^\mu)}_{\text{not invertable}} \cancel{=e}$$

Choose gauge $\partial_\mu A^\mu = 0 \Longrightarrow \Box A = J$

QFT  $J_\mu \frac{1}{q^2} J^\mu \xrightarrow[\text{NR FT}]{} e_1 \frac{1}{4\pi r} e_2$

Non abelian

$$\psi \to U\psi \longrightarrow J^i_\mu = \bar{\psi} \cdots \frac{\lambda^i}{2} \psi$$

$$D_\mu = \partial_\mu + ig \frac{\lambda^A}{2} A^A_\mu = \partial_\mu + ig \underline{A}_\mu$$

$$\underline{\underline{F}}_{\mu\nu} = \partial_\mu \underline{A}_\nu - \partial_\nu \underline{\underline{A}}_\mu + ig[\underline{A}_\mu, \underline{A}_\nu]$$

— ghosts

Gravity

Want energy as source
— Eötvös
— light bending
$\Big\} \Rightarrow T_{\mu\nu}$

$$H = \int d^3x \, T_{00}$$

$$\partial^\mu T_{\mu\nu} = 0$$

$T_{\mu\nu}$ is current for

$$x_\mu \to x_\mu + a_\mu$$

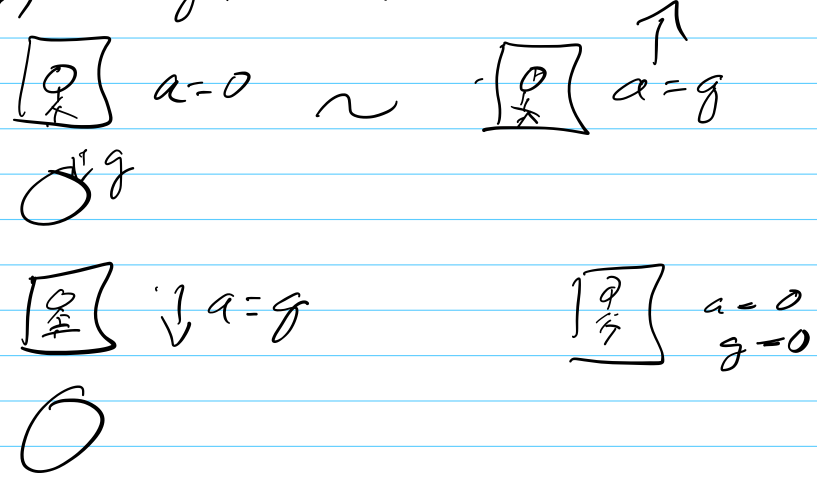
Ex: $\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi \, \partial^\mu \phi - m^2 \phi\right)$

$$T_{\mu\nu} = \partial_\mu \phi \, \partial_\nu \phi - \frac{\eta_{\mu\nu}}{2}\left(\partial_\sigma \phi \, \partial^\sigma \phi - m^2 \phi^2\right)$$

Gauge spacetime translation

$$\underline{\underline{x_\mu \to x_\mu + a_\mu(x)}} = x'_\mu(x)$$

Crazy May work!



~~New~~ New field

$$ds = \eta_{\mu\nu}\, dy^\mu\, dy^\nu \to g_{\mu\nu}(x)\, dx^\mu\, dx^\nu$$

$$dx'^\mu = J^\mu_{(x)\,\nu}\, dx^\nu \qquad \underbrace{J^\mu{}_\nu(x) = \frac{\partial x'^\mu}{\partial x^\nu}}$$

$$g'_{\mu\nu}\, dx'^\mu\, dx'^\nu = g_{\alpha\beta}\, dx^\alpha\, dx^\beta$$

$$dx' = J\, dx$$

$$g' = J^{-1}\, g\, J' \qquad g'_{\mu\nu} = J^{-1\,\alpha}_{\mu}\, J^{-1\,\beta}_{\nu}\, g_{\alpha\beta}$$

Success

$$S = \int \underbrace{d^4x\sqrt{-g}}_{\text{invariant}}\ \frac{1}{2}\left[g^{\mu\nu}\,\underline{\partial_\mu\phi\,\partial_\nu\phi} - m^2\phi^2\right]$$

$$\frac{\partial \mathcal{L}}{\partial g^{\mu\nu}} = \frac{\sqrt{-g}}{2}\left[\partial_\mu\phi\,\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}\left[\partial_\lambda\phi\,\partial^\lambda\phi - m^2\phi^2\right]\right]$$

$$= \frac{\sqrt{-g}}{2}\,\underline{T_{\mu\nu}}$$

Success #2

$$\overline{\underline{\ \ \mathcal{\xi} g\ \ }} = \frac{\kappa}{2}\,\overline{T_{\mu\nu}}\ \frac{P_{\mu\nu\alpha\beta}}{g^2}\ T^{\alpha\beta}$$

$$\overrightarrow{NR} \to \frac{\kappa^2}{32\pi}\ \frac{m_1 m_2}{r}$$

✓

Success #3

$$\frac{\partial S}{\partial \phi} = \underbrace{\frac{1}{\sqrt{-g}}\,\partial_\mu\left(g^{\mu\nu}\sqrt{-g}\,\partial_\nu\right)}_{D^2}\phi + m^2\phi = 0$$

$D_\mu \phi = \partial_\mu \phi$

$D_\mu(\partial_\nu\phi) = (\partial + \Gamma)\partial\phi$

N.R. reduction $\quad \phi(x,t) = \underline{e^{-imt}}\,\psi(x,t)$

small t dep.

Need $\quad g_{00} = 1 + 2\phi_g$

grav pot

$$\Longrightarrow \quad i\frac{\partial}{\partial t}\psi = \left[\frac{-\nabla^2}{2m} + m\phi_g\right]\psi$$

$m_I = m_g$

Force law $\quad \dot{\vec{p}} = -i\,[H, \vec{p}\,] = -m\nabla\phi = m\vec{a}$

Construct gravity part

$dx \to J\,dx$

$V^\mu \to \underline{D}V \to JJDV$

$DV = (\partial + \Gamma)V$

$$D_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^{\nu}\,V^\rho \qquad \partial_\mu + \underline{A}_\mu$$

Could be new field

or composed field $\quad \Gamma_{\mu\nu}{}^{\lambda} = \frac{1}{2}g^{\lambda\sigma}\left[\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu}\right]$

If use EP

$$\partial_\mu \eta_{\alpha\nu} = 0$$

$$\underbrace{D_\mu g_{\alpha\beta} = 0}_{0} \quad \leftarrow \text{metricity}$$

$$\Gamma = \Delta$$

Covariant

$$[D_\mu, D_\nu] V^\beta \qquad D'_\mu D'_\nu V^\beta$$
$$= R_{\mu\nu\alpha}{}^{\beta} V^\alpha \qquad \longrightarrow [\,][\,][\,][\,]$$

$$R_{\mu\nu\alpha}{}^{\beta} = \partial_\mu \Gamma_{\nu\alpha}{}^{\beta} - \partial_\nu \Gamma_{\mu\alpha}{}^{\beta} + \Gamma_{\mu\rho}^{\beta}\Gamma_{\nu}^{\rho} - \Gamma_{\nu\rho}^{\beta}\Gamma_{\mu}^{\rho}$$

$$R_{\nu\alpha} = R_{\mu\nu\alpha}{}^{\mu} \quad , \quad R = g^{\nu\alpha} R_{\nu\alpha}$$

$$g^{\mu\nu} g_{\nu\alpha} = \delta^\mu_\alpha$$

EFT $\quad \Gamma \Rightarrow \partial g$

$$R \sim \partial^2 g$$

Invariant action

$$S = \int d^4x \sqrt{-g}\left[-\Lambda - \frac{2}{\kappa^2} R + \underbrace{R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} \cdots}\right]$$

Eq. on Motion

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{\kappa^2}{4} T_{\mu\nu}$$

Weak field expansion

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}(x) \qquad g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \frac{1}{2} h^{\mu\lambda} h_\lambda^{\ \nu} + \cdots$$

$$\Big[\underbrace{-\Box h_{\mu\nu} - \partial_\mu \partial_\lambda h^\lambda_{\ \nu} \cdots}_{\text{no inverse}} \Big] = \frac{\kappa^2}{4} T_{\mu\nu}$$

$$O_{\mu\nu}{}^{\alpha\beta} G_{\alpha\beta\gamma\delta}(x-y) = \frac{1}{2} I_{\mu\nu\gamma\delta} \delta_D^{(4)}(x-y)\,,$$

where

$$O^{\mu\nu}{}_{\alpha\beta} \equiv (\delta_\alpha^{(\mu}\delta_\beta^{\nu)} - \eta^{\mu\nu}\eta_{\alpha\beta})\Box - 2\delta^{(\mu}_{(\alpha}\partial^{\nu)}\partial_{\beta)} + \eta_{\alpha\beta}\partial^\mu\partial^\nu + \eta^{\mu\nu}\partial_\alpha\partial_\beta\,.$$

Gauge invariance $\quad x^\mu \to x^\mu + \xi^\mu(x)$

$$h_{\mu\nu} \Rightarrow h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

Gauge fixing

harmonic gauge
de Donder gauge $\quad \left[\partial_\mu h^\mu_{\ \nu} - \frac{1}{2}\partial_\nu h^\lambda_{\ \lambda}\right] = 0$

Then

$$\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda_{\ \lambda}\right) = \frac{\kappa^2}{4} T_{\mu\nu}$$

Invertable:

$$D_{\mu\nu\alpha\beta} = \frac{1}{q^2}\,\frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} - \eta_{\mu\nu}\eta_{\alpha\beta}\right]$$

2nd quant.

$$h_{\mu\nu} = \sum \left[\epsilon^{\lambda}_{\mu\nu}\, a_{\lambda}\, e^{-ip\cdot x} + \dots \right]$$

$$\underline{\epsilon^{\lambda}_{\mu}\,\epsilon^{\lambda}_{\nu}}$$

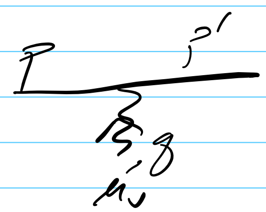
Point mass

$$h_{\mu\nu} = \begin{pmatrix} 2\phi_g & & & 0 \\ & 2\phi_g & & \\ & & 2\phi_g & \\ 0 & & & 2\phi_g \end{pmatrix} \qquad \phi_g = -\frac{GM}{r}$$

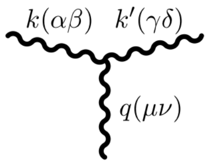
$$32\pi G = \kappa^2$$

Feynman rules

 $iD_{\mu\nu\alpha\beta}$



$$= \frac{\kappa}{2}\left[(p_\mu p'_\nu + p'_\mu p_\nu) - \eta_{\mu\nu}(p\cdot p' - m^2) \right]$$



$$
\begin{aligned}
&= \frac{i\kappa}{2}\Bigg(P_{\alpha\beta,\gamma\delta}\Big[k^\mu k^\nu + (k-q)^\mu (k-q)^\nu + q^\mu q^\nu - \frac{3}{2}\eta^{\mu\nu} q^2\Big] \\
&+ 2q_\lambda q_\sigma [I^{\lambda\sigma}{}_{\alpha\beta} I^{\mu\nu}{}_{\gamma\delta} + I^{\lambda\sigma}{}_{\gamma\delta} I^{\mu\nu}{}_{\alpha\beta} - I^{\lambda\mu}{}_{\alpha\beta} I^{\sigma\nu}{}_{\gamma\delta} - I^{\sigma\nu}{}_{\alpha\beta} I^{\lambda\nu}{}_{\gamma\delta}] \\
&+ [q_\lambda q^\mu (\eta_{\alpha\beta} I^{\lambda\nu}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\nu}{}_{\alpha\beta}) + q_\lambda q^\nu (\eta_{\alpha\beta} I^{\lambda\mu}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\lambda\mu}{}_{\alpha\beta}) \\
&- q^2 (\eta_{\alpha\beta} I^{\mu\nu}{}_{\gamma\delta} + \eta_{\gamma\delta} I^{\mu\nu}{}_{\alpha\beta}) - \eta^{\mu\nu} q^\lambda q^\sigma (\eta_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} + \eta_{\gamma\delta} I_{\alpha\beta,\lambda\sigma})] \\
&+ \Big[2q^\lambda \Big(I^{\sigma\nu}{}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k-q)^\mu + I^{\sigma\mu}{}_{\alpha\beta} I_{\gamma\delta,\lambda\sigma} (k-q)^\nu \\
&- I^{\sigma\nu}{}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\mu - I^{\sigma\mu}{}_{\gamma\delta} I_{\alpha\beta,\lambda\sigma} k^\nu \Big) \\
&+ q^2 (I^{\sigma\mu}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^\nu + I_{\alpha\beta,\sigma}{}^\nu I^{\sigma\mu}{}_{\alpha\delta}) + \eta^{\mu\nu} q^\lambda q_\sigma (I^{\rho\sigma}{}_{\gamma\delta} I_{\alpha\beta,\lambda\rho} + I^{\rho\sigma}{}_{\alpha\beta} I_{\gamma\delta,\lambda\rho})\Big] \\
&+ \Big[(k^2 + (k-q)^2)\left(I^{\sigma\mu}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^\nu + I^{\sigma\nu}{}_{\alpha\beta} I_{\gamma\delta,\sigma}{}^\mu - \frac{1}{2}\eta^{\mu\nu} P_{\alpha\beta,\gamma\delta}\right) \\
&- k^2 \eta_{\gamma\delta} I^{\mu\nu}{}_{\alpha\beta} - (k-q)^2 \eta_{\alpha\beta} I^{\mu\nu}{}_{\gamma\delta}\Big]\Bigg).
\end{aligned}
$$

← Ep of graviton

vanish on shell

$$I_{\mu\nu\alpha\beta} = \frac{1}{2}\left[\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta}\right]$$

<u>Reflect</u>

- GR from symmetry
  It is all fields.

$$\frac{2}{\kappa^2} R \longrightarrow -\frac{1}{4g^2} F^2$$

$$g = \eta + h$$

$$\longrightarrow \frac{1}{2}\partial h\, \partial h$$

QFT

 $\left(\frac{\kappa}{2}\right)^2 \frac{1}{2}\mu^{4-d}\int \frac{d^d k}{(2\pi)^d}\, \frac{1}{k^2}\,\frac{1}{(k-q)^2}\, 2\,[kkqq]$

$\uparrow 4k$

$$\underbrace{\sim \left[\frac{1}{\epsilon} - \ln q^2/\mu^2\right]\left[qqqq\right]}_{E^4}$$

Not like $R \sim \partial\partial \sim E^2$

$R^2 \sim \partial\partial\partial\partial \sim E^4$

$$\Longrightarrow \Delta\mathcal{L} = \frac{1}{\epsilon}\left(aR^2 + bR_{\mu\nu}R^{\mu\nu}\right)$$

small q

R^2 terms will be generated in S_g

$$N'_\mu = A_\mu^{\ \nu}(x)\, N_\nu + a_\mu(x)$$