UV complete 4-derivative scalar field theory

Bob Holdom

UNIVERSITY OF TORONTO

Puzzles in the Quantum Gravity Landscape
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the QFT

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi(\Box + m^2)\partial^\mu \phi + \lambda_3(\partial_\mu \phi \partial^\mu \phi)\Box\phi + \lambda_4(\partial_\mu \phi \partial^\mu \phi)^2$$

- 4-derivatives, both in the interaction terms and the kinetic terms
- dimensionless real scalar field $\phi(x)$ and dimensionless couplings λ_3 and λ_4
- shift symmetry $\phi \rightarrow \phi + c$
- m^2 breaks the classical scale invariance

proxy for quadratic gravity

- Einstein action is supplemented with terms quadratic in curvature, and these terms bring in 4-derivatives
- 4-derivatives in kinetic and interaction terms
- both theories are renormalizable
- the shift symmetry is playing the role of coordinate invariance of the gravity theory
- the $m^2\partial_\mu\phi\partial^\mu\phi$ is playing the role of the Einstein term
- at low energies this term dominates; left with a normal massless field with non-renormalizable interactions

UV completeness

- both theories are UV complete, and so we can see what happens at energies much higher than m
- also refer to this as the $m \to 0$ limit
- ultra-Planckian energies in the case of gravity
- the story turns out to be very similar for the two theories, since it is really just about the physics of four derivatives
- scalar theory is easier to deal with, so will focus on that
- return to quadratic gravity at the end

ghost

- propagator has massive pole with abnormal sign residue
- the negative norm state is said to be in conflict with unitarity
- what is actually meant by this is that the theory may have problems with positivity
- S-matrix unitarity can still be defined in the presence of negative norm states
- $S\mathbb{1}S^\dagger = \mathbb{1}$ where $\mathbb{1} = \sum_X \frac{|X\rangle\langle X|}{\langle X|X\rangle}$ reflects the negative norms
- S-matrix unitarity means that probability is conserved

optical theorem

- also the optical theorem can be directly verified in perturbation theory by keeping track of minus signs
- the LHS is imag part of forward scattering amplitude, and its calculation is affected by any wrong-sign propagators
- the RHS is a scattering process into on-shell final states, and this is affected by any negative norms among these states
- it can thus be seen that the LHS and RHS of the optical theorem are both affected in such a way that it remains satisfied

positivity

- if the issue is positivity rather than unitarity then this at least leaves some room open for discussion
- there are certainly some abnormal minus signs floating around in calculations, but the question is whether physical quantities that should be positive, can end up being positive
- this needs some investigation
- our focus will be on the positivity constraint in the high energy limit, but we will return to lower energies

β-functions

- renormalization of $\partial_\mu \phi \Box \partial^\mu \phi$ term is treated as a standard wave function renormalization

$$\frac{d\lambda_3}{d\ln\mu} = -\frac{5}{4\pi^2}(\lambda_4\lambda_3 + \frac{3}{4}\lambda_3^3)$$

$$\frac{d\lambda_4}{d\ln\mu} = -\frac{5}{4\pi^2}(\lambda_4^2 + \lambda_4\lambda_3^2)$$

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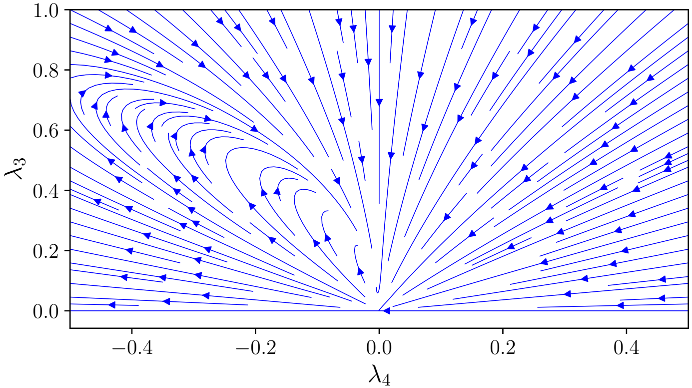
renormalization group flow



- arrows point to the UV
- asymptotic freedom in UV
- some flows also show asymptotic freedom in IR

a new mass scale

- flow towards IR stops when the energy scale drops below m; this is the transition to the low energy theory
- the crossover to this low energy theory may occur at weak couplings, in which case the theory remains perturbative at all scales
- but for sufficiently small m, the flow towards the IR can result in large couplings
- creates a new mass scale through dimensional transmutation
- it is this that can be the origin of Planck mass in gravity

outline

- describe a simplified method for calculating in the high energy limit
- old method involves decomposing ϕ into two degrees of freedom
- in new method there appears to be only one degree of freedom at high energies
- calculate the optical theorem and a differential cross section as functions of λ_3 and λ_4
- this gives expressions that we can test for positivity
- positivity picks out the allowed region on the RG flow plane

a related issue

- four derivative interaction terms apparently produce diverging amplitudes at large momenta
- nonstandard cancellations take place at the cross section level (also in quadratic gravity)
- the reduction to effectively one degree of freedom at high energies clarifies what is happening
- makes more clear the origin of good high energy behaviour

mass derivative

- four derivative propagator $G^{(4)}(p^2, m^2)$ can be written in terms of the Feynman propagator

$$G^{(2)}(p^2, m^2) = \frac{1}{p^2 - m^2 + i\varepsilon}$$

as

$$G^{(4)}(p^2, m^2) = -\frac{G^{(2)}(p^2, m^2) - G^{(2)}(p^2, 0)}{m^2}$$

- thus in the $m \to 0$ limit (high energy limit)

$$\lim_{m \to 0} G^{(4)}(p^2, m^2) = \lim_{m \to 0} (-\frac{d}{dm^2}) G^{(2)}(p^2, m^2)$$

mass derivative in optical theorem

- the imaginary part of a forward scattering amplitude $\mathcal{A}_{i\to i}$ is extracted by cutting propagators and using

$$\mathrm{Im}(G^{(2)}(p^2, m^2)) = -i\pi\delta(p^2 - m^2).$$

- the analog for $G^{(4)}$ in the $m \to 0$ limit is

$$\lim_{m\to 0} \mathrm{Im}(G^{(4)}(p^2, m^2)) = -i\pi \lim_{m\to 0}(-\frac{d}{dm^2})\delta(p^2 - m^2)$$

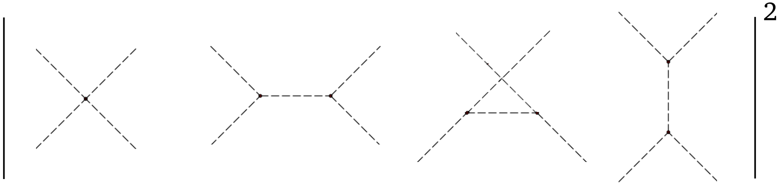
- the additional operation, $-\lim_{m\to 0}\frac{d}{dm^2}$, also works on the RHS of the optical theorem

RHS of optical theorem

- each on-shell particle in a final state f should be assigned its own dummy mass m_j
- $|\mathcal{A}_{i\to f}|^2$ will depend on the values of these m_j's via the on-shell conditions
- take the mass derivatives, then the $m_j \to 0$ limits, then the phase space integral
- term on the RHS of the optical theorem corresponding to the final state f takes the form

$$\lim_{m_j\to 0}\left(\prod_{j=1}^{n}(-\frac{d}{dm_j^2})|\mathcal{A}_{i\to f}(m_1..m_n)|^2\right)$$

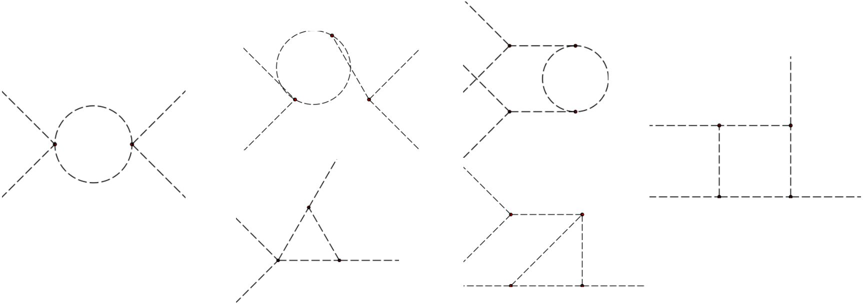
$|\mathcal{A}_{i\to f}(m_1..m_n)|^2$ for $\phi\phi \to \phi\phi$



- new method reproduces the usual sum over the $\phi\phi$ final states
  - in that sum, for each ϕ, choose one mass, then the other, insert a minus sign for the negative norm, and divide by m^2, due to the way field is normalized

now the LHS of optical theorem

- imaginary part of the forward scattering amplitude
- various diagrams are of order λ_4^2, $\lambda_4 \lambda_3^2$ or λ_3^4



calculate diagrams for LHS

- can again use mass derivatives and $m_j \to 0$
- limit is smooth since the imaginary part has no infrared divergences
- can thus use the massless $G^{(4)}$ propagator directly

$$G^{(4)}(p^2, 0) = -\frac{1}{(p^2 + i\varepsilon)^2}$$

- square of a Feynman propagator can be handled by standard methods for calculating one-loop diagrams
- 4-derivative vertices still complicate the calculation

optical theorem for $\phi\phi \to \phi\phi$

- calculate the two sides of the optical theorem independently

$$\text{LHS} = \text{RHS} = \frac{s^2}{6\pi}(6\lambda_3^4 + 19\lambda_3^2\lambda_4 + 14\lambda_4^2)$$

- the equality demonstrates S-matrix unitarity

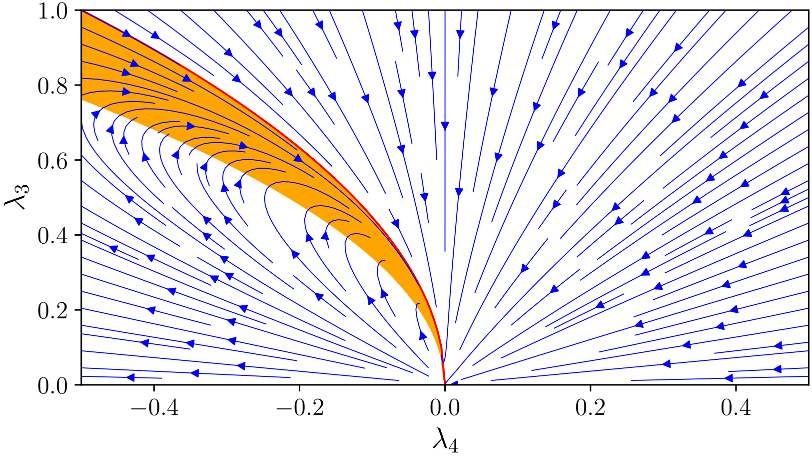
- both sides calculated without decomposing ϕ field into positive and negative norm parts

- RHS naively goes like s^4, being the square of amplitudes that go like s^2

- is reduced to s^2 behaviour because of $-\frac{d}{dm^2}$ applied twice

the positivity constraint

- LHS = RHS is negative for $-\frac{6}{7} < \lambda_4/\lambda_3^2 < -\frac{1}{2}$
- this region is shaded orange, and the red line is $\lambda_4 = -\frac{1}{2}\lambda_3^2$



meaning of red line

- the red line marks the boundary between two sets of flows that are qualitatively different
- the flows below this line will eventually enter the orange region in the UV
- thus all such flows are forbidden
- the allowed flows are on or to the right of the red line
- these couplings are asymptotically free in the UV and can become strong in the IR

two degrees of freedom?

- consider the two fields constructed from ϕ,

$$\psi_1 = \frac{1}{m^2}(\Box + m^2)\phi$$
$$\psi_2 = \frac{1}{m^2}\Box\phi$$

- when expressed in terms of ψ_1 and ψ_2 the kinetic term of the Lagrangian becomes

$$-\frac{m^2}{2}\psi_1\Box\psi_1 + \frac{m^2}{2}\psi_2(\Box + m^2)\psi_2$$

- ψ_1 and ψ_2 are the two fields of definite mass (0 and m) and definite norm (+ and $-$)
- but we also see that $\phi = \psi_1 - \psi_2$

one degree of freedom

- $\phi = \psi_1 - \psi_2$ is the only combination that appears in all the interaction terms

- we have introduced the operation $-\lim_{m\to 0} \frac{d}{dm^2}$ for every external ϕ line, and this also treats ψ_1 and ψ_2 on equal footing with a relative minus sign

- the external state matches the interacting state

- the apparent two degrees of freedom have been reduced to one

differential cross section for $\phi\phi \to \phi\phi$

- treat it as a single process and take four mass derivatives
- need the dependence on the set of four masses m_j
- before taking mass derivatives, would diverge as $\sim (s^2)^2/s$ for large s
- a term $\sim m_1^2 m_2^2 m_3^2 m_4^2/s$ is needed to survive four m_j^2-derivatives and $m_j \to 0$
- in the end we have a differential cross section that behaves like $1/s$ at large s times a function of the scattering angle
- reason for good high energy behaviour is now clear

result

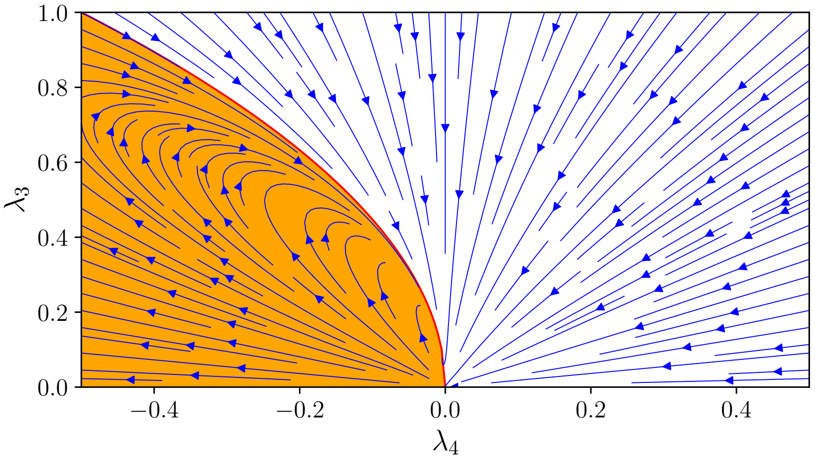
- the differential cross section for $\phi\phi \to \phi\phi$ scattering at high energies

$$\frac{d\sigma}{d\Omega} = \Big(\left(\lambda_3^4 - 4\lambda_4^2\right)\sin(\theta)^6 + 24\lambda_4^2\sin(\theta)^4 + \left(-48\lambda_3^4 - 96\lambda_3^2\lambda_4\right)\sin(\theta)^2 \\ +64\lambda_3^4 + 128\lambda_3^2\lambda_4\Big) / (16\pi^2\sin(\theta)^4 s)$$

- this result is positive definite for any θ as long as $\lambda_4 \geq -\frac{1}{2}\lambda_3^2$
- this is the same constraint as before!
- interesting special case on the red line:

$$\frac{d\sigma}{d\Omega} = \frac{3\lambda_4^2}{2\pi^2 s} \quad \text{when } \lambda_4 = -\frac{1}{2}\lambda_3^2$$

the $\lambda_4 \geq -\frac{1}{2}\lambda_3^2$ constraint



- positivity has picked out the running couplings that flow to strong coupling in the infrared

QCD analogy

- we have been able to work in the asymptotically free regime where the perturbative degrees of freedom are appropriate
- this is similar to using perturbative QCD to study scattering of quarks and gluons at very high energies, even though quarks and gluons are not the asymptotic states
- they are not because of strong interactions at intermediate energies
- so what are the asymptotic states in the 4-derivative theory?

asymptotic states

- with strong interactions, the poles of the bare propagators need not correspond to true asymptotic states of the theory
- another example, the sigma meson — has a width of order its mass ~ 0.5 GeV — and is in no way an asymptotic state
- for both the scalar theory and the gravity theory, we need to suppose that the ghost is not an asymptotic state
- as long as asymptotic states have positive norm, all probabilities are positive
- virtual effects of the ghost are okay

shift symmetry

- the massless field $\psi_1 = \frac{1}{m^2}(\Box + m^2)\phi$ transforms under the shift symmetry $\psi_1 \to \psi_1 + c$ in the same way as ϕ does
- we can suppose that this shift symmetry is not broken by the strong interactions
- then ψ_1 survives as a true asymptotic state
- we end up with one degree of freedom, at whatever energy scale is used to probe the theory
- this is either ψ_1, or $\psi_1 - \psi_2$ in the perturbative high energy description

quadratic gravity

$$S = -\frac{1}{16\pi G}\int d^4x\sqrt{-g}\left(R + \frac{R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2}{m_G^2} - \frac{R^2}{6m_S^2}\right)$$

- introduces massive spin-2 ghost and massive scalar
- Gm_G^2 is a dimensionless and asymptotically free coupling
- spin-2 sector has graviton and ghost — completely analogous to ψ_1 and ψ_2
- our comments about ψ_1 and ψ_2 also apply to the graviton and the ghost — except that the shift symmetry is replaced by coordinate invariance

ghost summary

- in high energy limit the ghost is entering on an equal footing as a perturbative degree of freedom
- it plays an intrinsic role in achieving positivity and well-behaved cross sections
- this high energy picture is independent of whether or not the ghost is an asymptotic state
- but in the full theory, not being an asymptotic state seems to be required

quadratic gravity gives change of perspective

- on very small — standard QFT picture at ultra-Planckian energies
- on very large — arbitrarily large, horizonless, classical solutions, called 2-2-holes

what is a 2-2-hole?

- gravitationally bound ball of relativistic gas
- compactness essentially the same as a black hole
- integrate the entropy density of the gas to get the total entropy S_{22}

$$T_\infty S_{22} = T_{\mathrm{BH}} S_{\mathrm{BH}} = \frac{M}{2}$$

$$\frac{S_{22}}{S_{\mathrm{BH}}} = 0.7548 N^{\frac{1}{4}} \left(\frac{m_G}{m_{\mathrm{Pl}}} \right)^{\frac{1}{2}} \gtrsim 1$$

- N is number of species, $S_{22} \propto N^{\frac{1}{4}}$ is a feature, not a bug

position space propagator

- standard position space propagator $G^{(2)}(x^2, m^2)$

$$G^{(2)} = \frac{-im\theta(x^2)}{4\pi^2\sqrt{x^2}}K_1(im\sqrt{x^2}) + \frac{m\theta(-x^2)}{4\pi^2\sqrt{-x^2}}K_1(m\sqrt{-x^2}) + \frac{i}{4\pi^2}\delta(x^2)$$

- taking the mass derivative and then $m\sqrt{x^2} \to 0$

$$\begin{aligned} G^{(4)} &= -\frac{\theta(x^2)}{8\pi^2}K_0(im\sqrt{x^2}) - \frac{\theta(-x^2)}{8\pi^2}K_0(m\sqrt{-x^2}) \\ &\approx \frac{\theta(x^2)}{8\pi^2}\left(\log(m\sqrt{x^2}) + \frac{i\pi}{2}\right) + \frac{\theta(-x^2)}{8\pi^2}\log(m\sqrt{-x^2}) \end{aligned}$$

- gives independent boundary condition on the full propagator