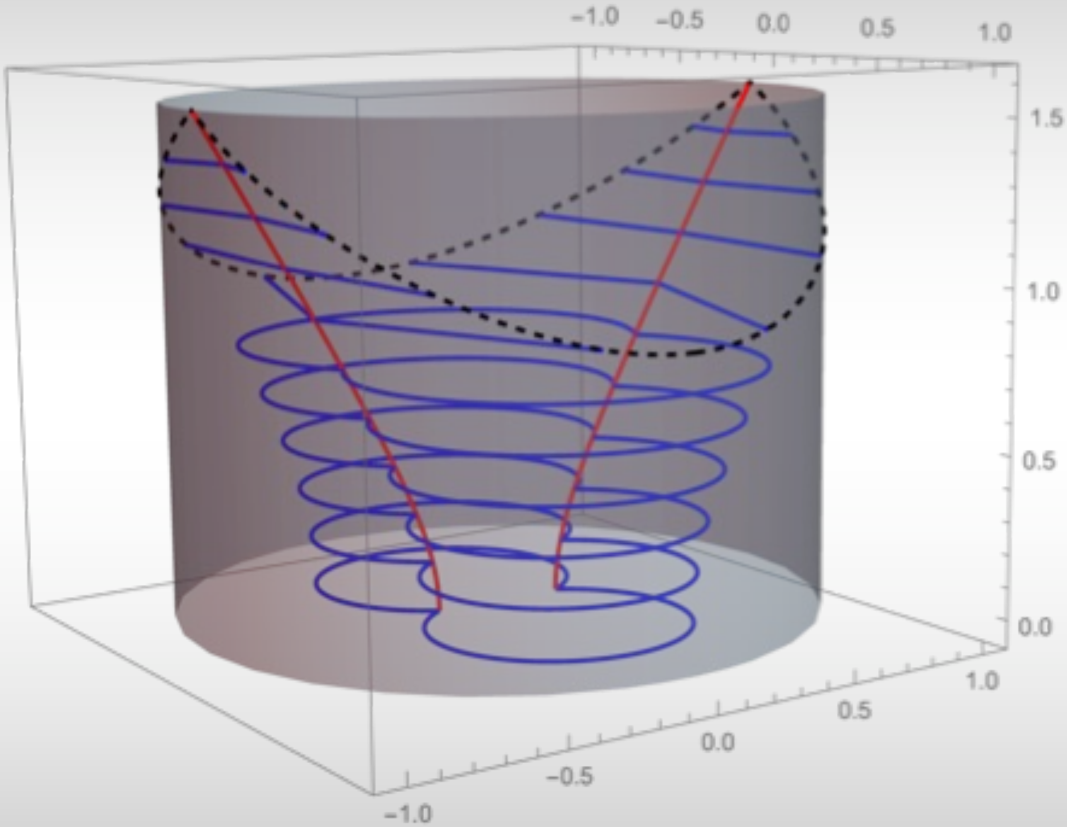
Holographic Thermodynamics: C-ing a Black Hole



Ruth Gregory

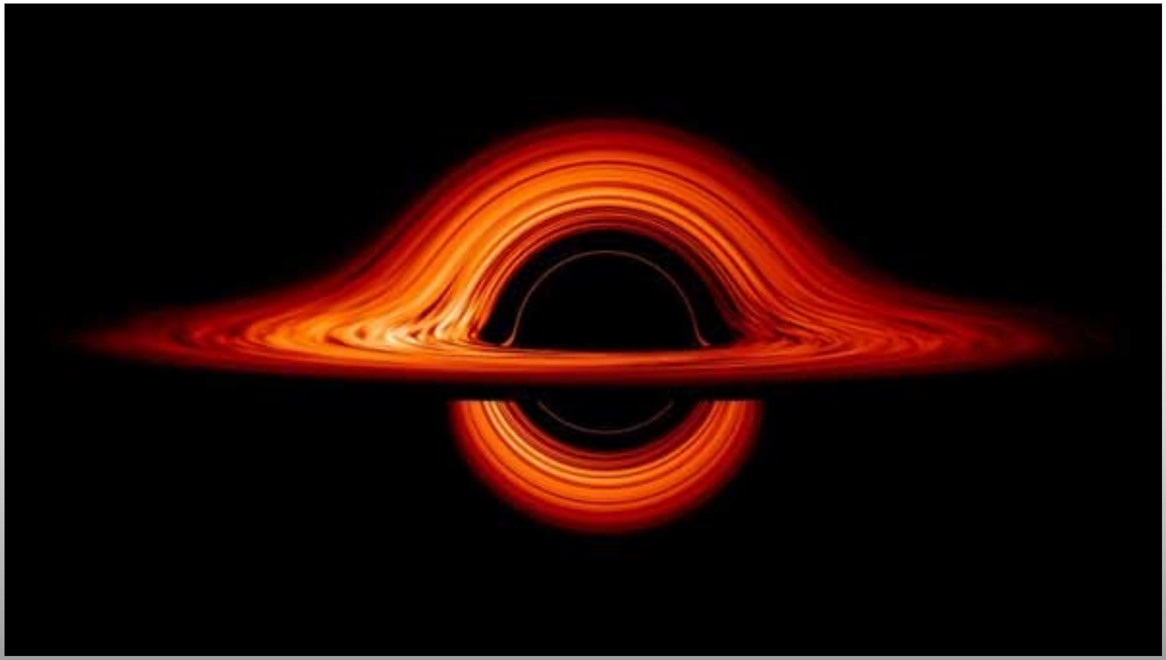
PI: Puzzles in the Quantum Gravity Landscape

26/10/23

Andres Anabalon, Gabriel Arenas-Henriquez, Mike Appels, Adolfo Cisterna, Felipe Diaz, Finn Gray, David Kastor, David Kubiznak, Zheng Liang Lim, Rob Mann, Ali Ovgun, Andy Scoins, Jennie Traschen

Black Holes

Are endlessly fascinating! Manifestation of strong gravity, and hold clues to quantisation of spacetime. Black holes thermodynamics is an odd mix of classical techniques & quantum implications.



"Classical" BH Thermodynamics

Consider the Schwarzschild-AdS metric:

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2 d\Omega_{II}^2 \qquad f(r) = 1 - \frac{2m}{r} + \frac{r^2}{\ell^2}$$

Black hole horizon defined by f=0, look at small changes in f. Horizon still defined by f(r) = 0.

$$f(r_+ + \delta r_+) = f'(r_+)\delta r_+ + \frac{\partial f}{\partial m}\delta m + \frac{\partial f}{\partial \ell}\delta \ell = 0$$

Changes r+, hence S

Changes m, hence M

Changes ℓ, hence Λ

Classical BH Thermodynamics

The entropy is given by the area of the horizon, and we usually calculate the temperature by Euclideanisation:

$$\delta S = 8\pi r_+ \delta r_+ \qquad T = f'(r_+)/4\pi$$

The cosmological "constant" defines a pressure term:

$$P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi\ell^2} \qquad \frac{\partial f}{\partial \ell}\delta\ell = -\frac{2r_+^2}{\ell^3}\delta\ell = \frac{8\pi r_+^2}{3}\delta P$$

Hence we end up with a First Law

$$\delta m = T\delta S + V\delta P$$

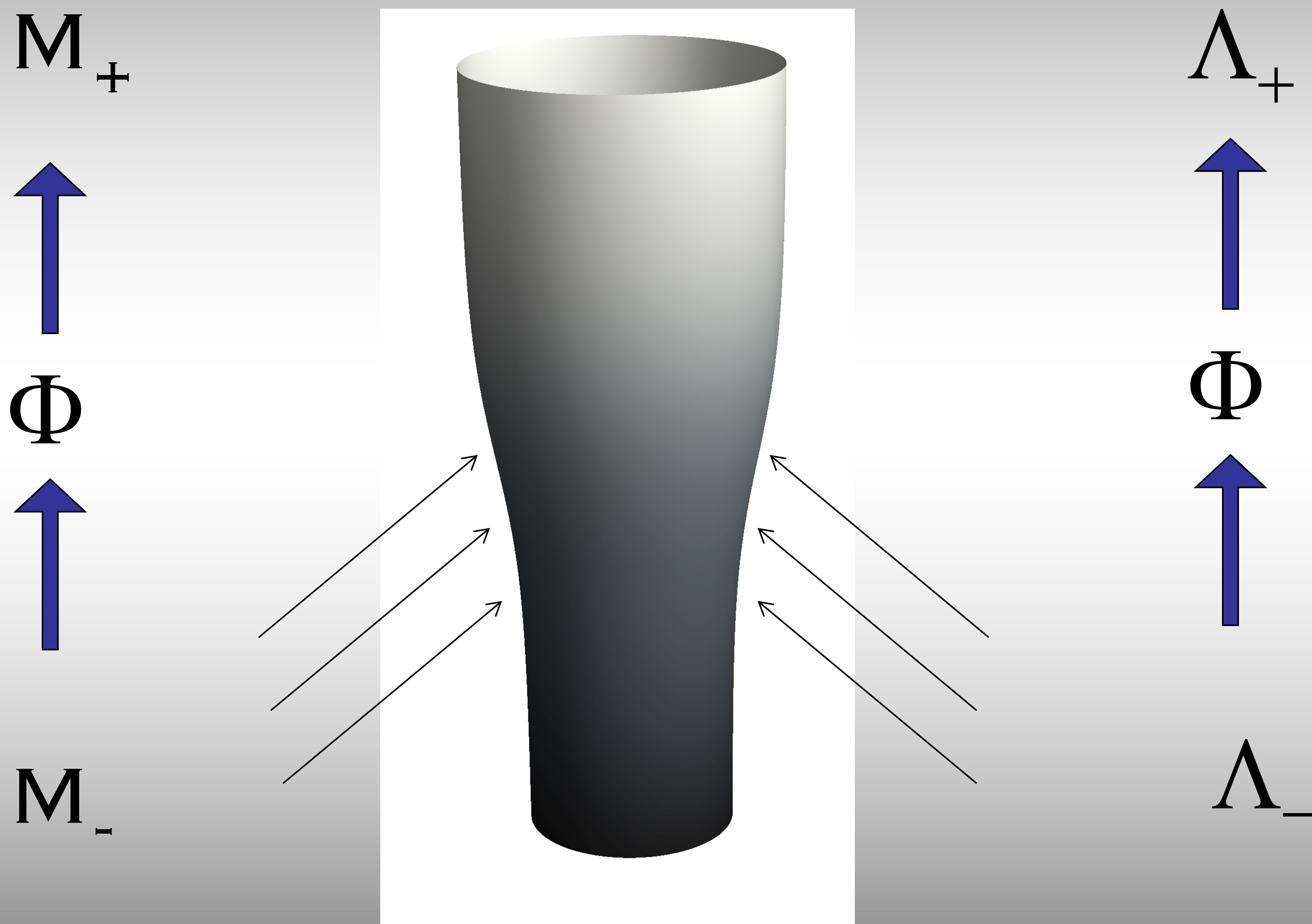
(identifying m with the enthalpy).

BH Thermodynamics

But it's interesting to dig deeper…

- ❖ What does dP mean?
- ❖ **What about composite systems?**
- ❖ Is our First Law unique?
- ❖ How do we know what dM is?

dP: or, How to vary Lambda:



Thermodynamics

The Einstein equations in null gauge become exact on the event horizons, so can find exact, differential forms of the various thermodynamic first laws:

- De Sitter patch:

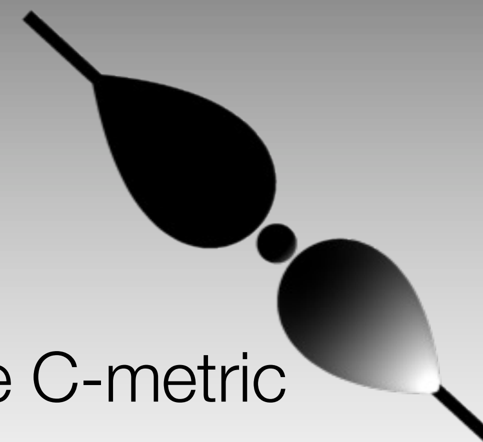
$$|\kappa_b|\dot{A}_b + |\kappa_c|\dot{A}_c + V\dot{\Lambda} = 0$$

- Black hole first law:

$$\frac{\dot{M}}{M_p^2} - |\kappa_b|\dot{A}_b + V_b\dot{\Lambda} = 0$$

Slow Roll with Black Holes – RG, D. Kastor and J. Traschen: 1707.06586, 1804.03462

Composite Systems: Accelerating Black Holes



An accelerating black hole is described by the C-metric

$$ds^2 = \Omega^{-2}\left[f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\left(\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\phi^2}{K^2}\right)\right]$$

Where

$$f = \left(1 - \frac{2m}{r}\right)(1 - A^2r^2) + \frac{r^2}{\ell^2}$$

$$g = 1 + 2mA\cos\theta$$

$$\Omega = 1 + Ar\cos\theta$$

f determines horizon structure – black hole / acceleration / cosmological constant

Hong & Teo, CQG20 3629 (2003)

Conical Deficits

First, focus on the conical defect, K:

Isolating the effect of K (A=0) look on axis through black hole:

$$ds^2_{\theta,\phi} \propto d\theta^2 + \frac{\theta^2}{K^2} d\phi^2$$

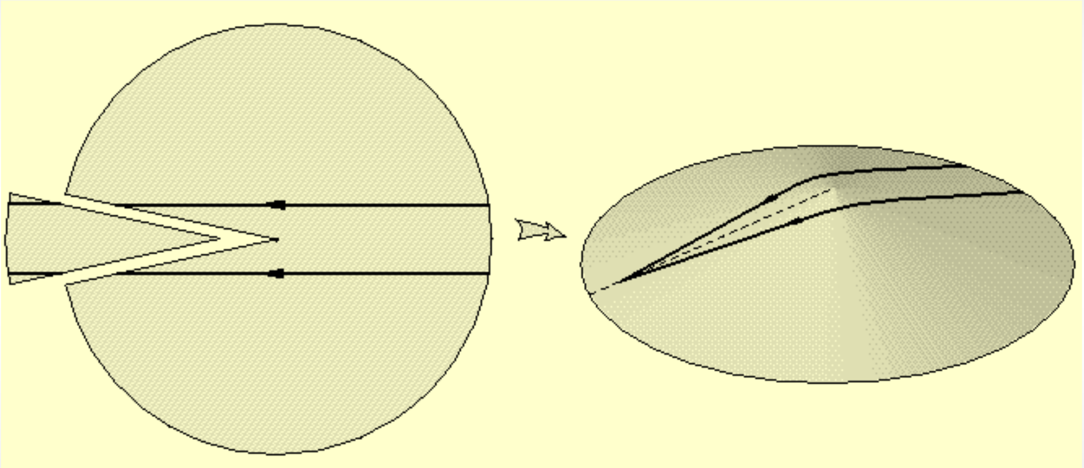
K relates to tension of "cosmic string" on axis

$$\delta = 2\pi \left(1 - \frac{1}{K}\right) = 8\pi\mu$$

c.f. Aryal, Ford, Vilenkin: PRD 34, 2263 (1986), Achucarro, Gregory, Kuijken: PRD 52 5729 (1995)

Cosmic String

$$T^{\mu}_{\nu} \approx \delta^{(2)}(\mathbf{r})\ \mathrm{diag}\,(\mu, \mu, 0, 0)$$



A string produces a conical deficit, but no long range spacetime curvature (no tidal forces).

$$\delta = 8\pi G\mu$$

Thermodynamics with Strings

Return to the Schwarzschild-AdS metric, but with a deficit

$$ds^2 = f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\left[d\theta^2 + \sin^2\theta\frac{d\phi^2}{K^2}\right]$$

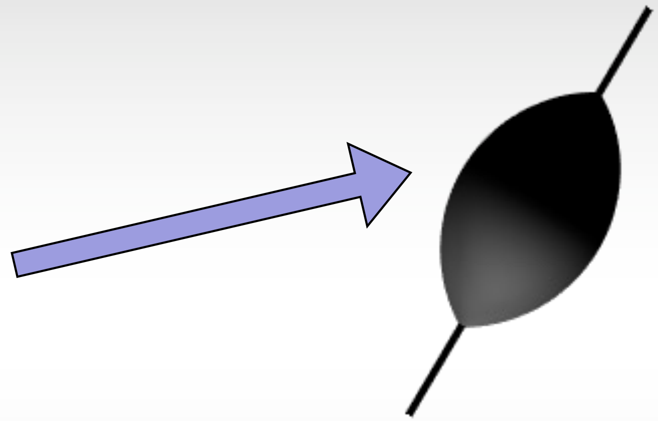
Still have the same relation:

$$f(r_+ + \delta r_+) = f'(r_+)\delta r_+ - \frac{2\delta m}{r_+} - \frac{r_+^2}{\ell^3}\delta\ell = 0$$

Entropy with Strings

But while temperature has the same expression, the entropy has changed:

$$S = \frac{\pi r_+^2}{K}$$



So

$$f'(r_+)\delta r_+ = \frac{2K}{r_+}\left(T\delta S + \frac{r_+^2 f'(r_+)}{4}\frac{\delta K}{K^2}\right)$$

And we have to identify changes in K

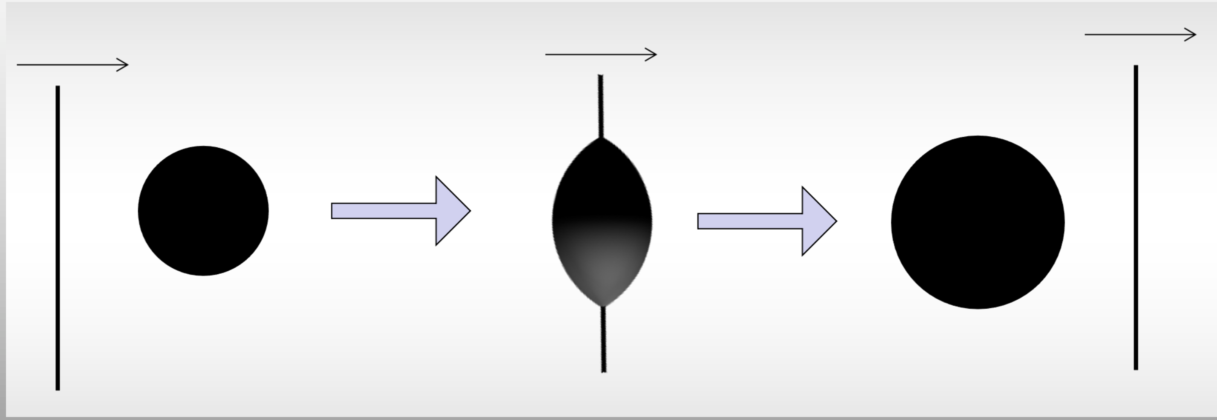
(S): Herdeiro, Kleihaus,Kunz,Radu: 0912:3386 [gr-qc]

Changing Tension

Tension is related to K: $$\mu = \frac{1}{4}\left(1 - \frac{1}{K}\right)$$

So easily get $$\delta\mu = \frac{\delta K}{4K^2}$$

Finally $$P = -\Lambda = \frac{3}{8\pi\ell^2} \qquad V = \frac{4\pi r_+^3}{3K}$$



First Law with Tension

Putting together:

$$0 = \frac{2K}{r_+}\left(T\delta S + 2(m - r_+)\delta\mu + V\delta P - \delta(\frac{m}{K})\right)$$

So identify

$$M = \frac{m}{K}$$

Then also get Smarr relation:

$$M = 2TS - 2PV$$

Thermodynamic Length

The term multiplying the variation in tension is a "thermodynamic length"

$$\lambda = r_+ - m$$

Reinforces interpretation of M as **enthalpy**, if black hole grows, it swallows some string, but has also displaced the same amount of energy from environment.

Kastor & Traschen: 1207:5415 [hep-th]

Quick Recap

Here, we have derived thermodynamics of a non-isolated black holes – a black hole threaded by a cosmic string.

The string happily joins in with the thermodynamic game and has its own thermodynamic charge and potential.

The metric has three free parameters: $m \quad \ell \quad K$

and we have three charges: $M \quad P \quad \mu$

Why spell this out? – Full cohomogeneity

On Acceleration

Acceleration is when an object is not travelling on a geodesic.

$$\nabla_T T \not\propto T$$

For an observer at $R=R_0$ in AdS:

$$ds^2_{AdS} = \left(1 + \frac{R^2}{\ell^2}\right)dt^2 - \frac{dR^2}{1+\frac{R^2}{\ell^2}} - R^2\left(d\Theta^2 + \sin^2\Theta d\phi^2\right)$$

The tangent vector is purely timelike, but the acceleration is radial:

$$\mathbf{T} = \frac{1}{\sqrt{1+\frac{R_0^2}{\ell^2}}}\frac{\partial}{\partial t} \qquad \mathbf{A} = \nabla_T T = \frac{R_0}{\ell^2}\frac{\partial}{\partial r}$$

Rindler with no Horizon

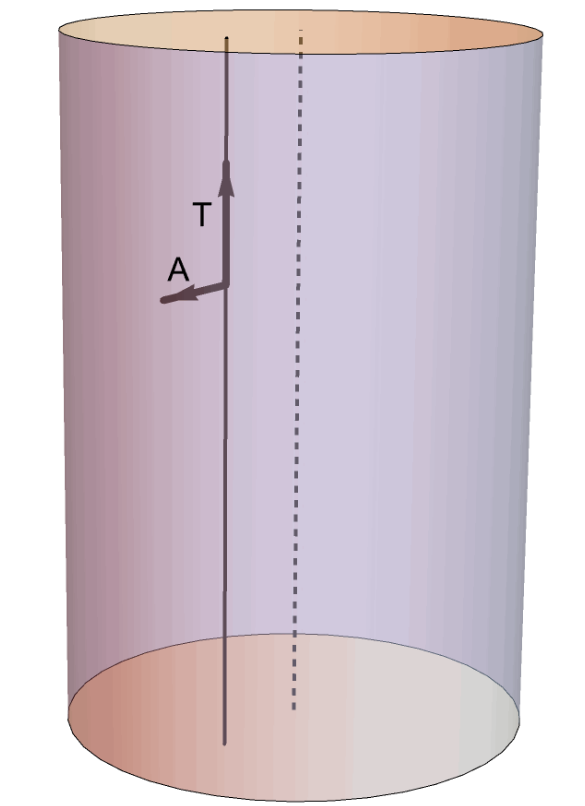
The magnitude of the acceleration is related to R_0

$$\mathbf{T} = \frac{1}{\sqrt{1+\frac{R_0^2}{\ell^2}}}\frac{\partial}{\partial t}$$

$$|\mathbf{A}|^2 = \frac{R_0^2/\ell^4}{1+R_0^2/\ell^2}$$

$$\mathbf{A} = \frac{R_0}{\ell^2}\frac{\partial}{\partial r}$$

$$R_0 = \frac{A\ell^2}{\sqrt{1-A^2\ell^2}}$$



Slowly Accelerating Rindler

Set m to zero:

$$ds^2 = \frac{\left[f(r)\frac{dt^2}{\alpha^2} - \frac{dr^2}{f(r)} - r^2 d\Omega_{II}^2\right]}{(1 + Ar\cos\theta)^2} \qquad f(r) = 1 + \frac{r^2(1 - A^2\ell^2)}{\ell^2}$$

Then use the coordinate transformation

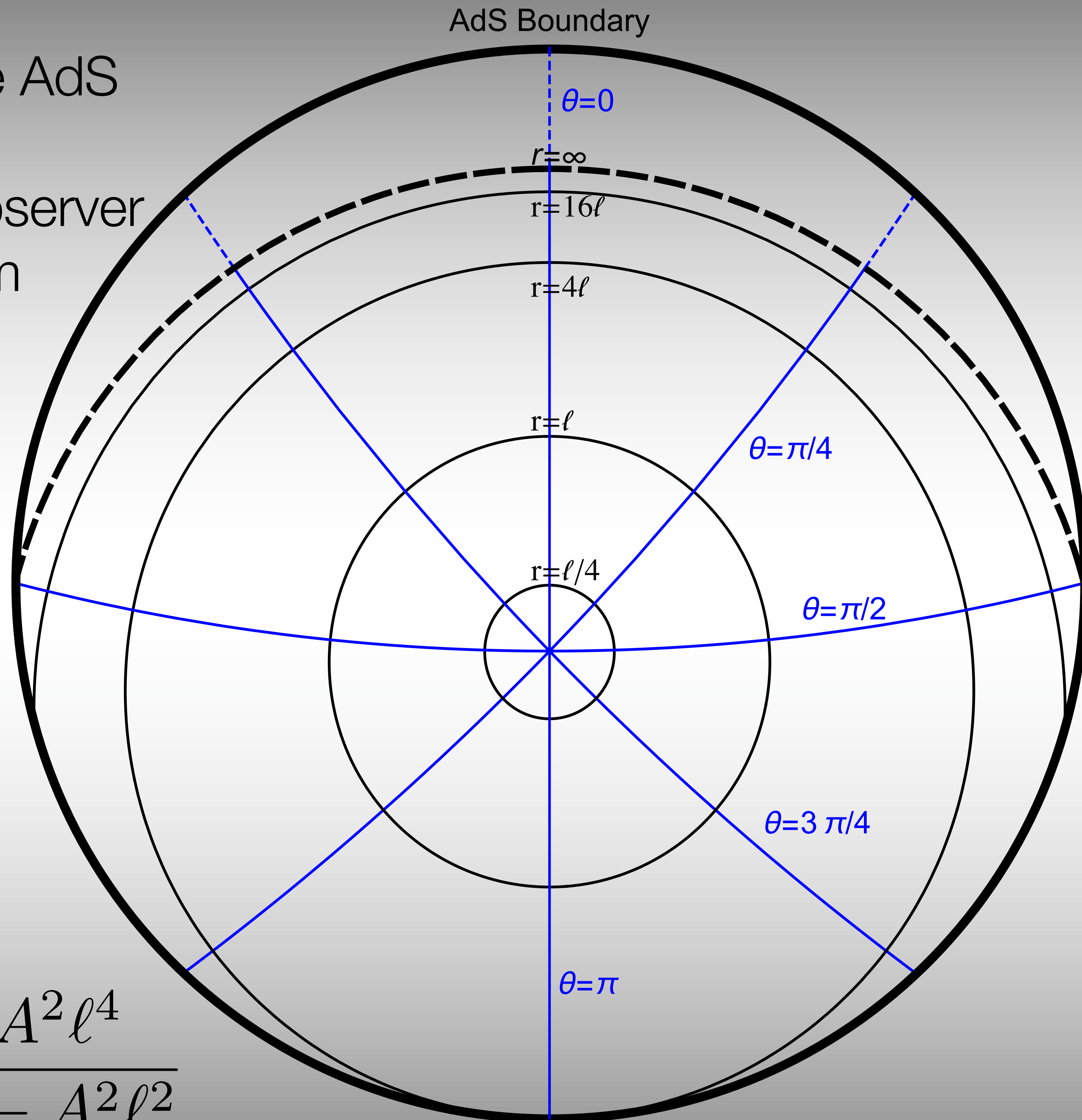
$$1 + \frac{R^2}{\ell^2} = \frac{1 + (1 - A^2\ell^2)r^2/\ell^2}{(1 - A^2\ell^2)\Omega^2}, \qquad R\sin\Theta = \frac{r\sin\theta}{\Omega}$$

to get back to global AdS

$$ds^2_{AdS} = \left(1 + \frac{R^2}{\ell^2}\right)dt^2 - \frac{dR^2}{1 + \frac{R^2}{\ell^2}} - R^2\left(d\Theta^2 + \sin^2\Theta\frac{d\phi^2}{K^2}\right)$$

$$\alpha^2 = 1 - A^2\ell^2$$

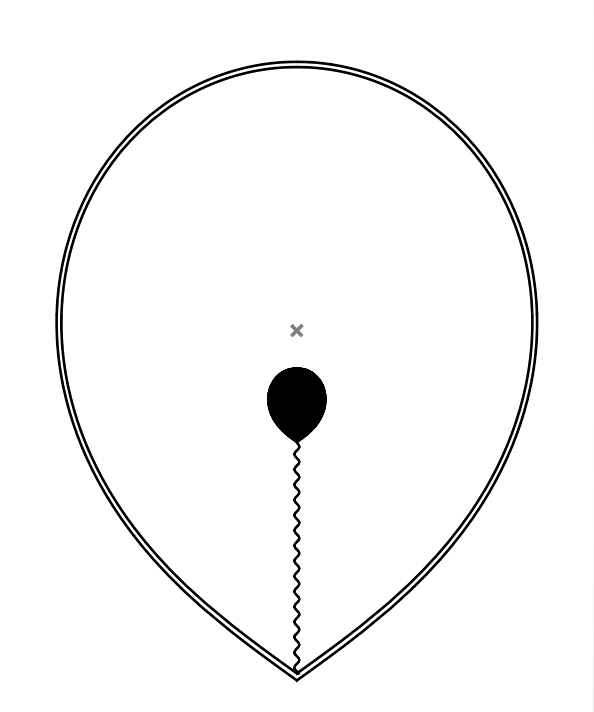
C-coordinates give AdS from an off-centre perspective. An observer hovering away from centre of AdS is accelerating.



$$r = 0 \leftrightarrow R = \frac{A^2\ell^4}{1 - A^2\ell^2}$$

The Slowly Accelerating Black Hole

The slowly accelerating black hole in AdS is displaced from centre. It has a conical deficit running from the horizon to the boundary. The string tension provides the force that hold the black hole off-centre.



Acceleration and Black Hole

Putting together, A gives the acceleration of the black hole, driven by an imbalance between North and South axes that now have different conical deficits.

- $\theta \to 0$ $$ds^2_{\theta,\phi} \propto d\theta^2 + \frac{(1+2mA)^2}{K^2}\theta^2 d\phi^2$$
- $\theta \to \pi$ $$ds^2_{\theta,\phi} \propto d\theta^2 + \frac{(1-2mA)^2}{K^2}(\pi-\theta)^2 d\phi^2$$

$$\delta_\pm = 2\pi\Big(1 - \frac{g(0)}{K}\Big) = 2\pi\Big(1 - \frac{1 \pm 2mA}{K}\Big) = \text{“}8\pi\mu_\pm\text{”}$$

We often make N axis regular, with deficit on S axis

$$\delta_N = 0 \Rightarrow K = 1 + 2mA \qquad \delta_S = \frac{8\pi mA}{K} \Rightarrow \mu_S = \frac{mA}{K}$$

(but this is a constrained system)

The Full Monty:

Based on experience with the Kerr-AdS metric (and motivated by the coordinate transformation for slowly accelerating Rindler) we have a possible rescaling of the time coordinate

$$ds^2 = \frac{1}{H^2}\left\{ \frac{f(r)}{\Sigma}\left[\frac{dt}{\alpha} - a\sin^2\theta\frac{d\varphi}{K}\right]^2 - \frac{\Sigma}{f(r)}dr^2 - \frac{\Sigma r^2}{h(\theta)}d\theta^2 - \frac{h(\theta)\sin^2\theta}{\Sigma r^2}\left[\frac{a\,dt}{\alpha} - (r^2+a^2)\frac{d\varphi}{K}\right]^2\right\}$$

This will rescale temperature, and also changes computations of the mass.

$$f(r) = (1 - A^2 r^2)\left[1 - \frac{2m}{r} + \frac{a^2+e^2}{r^2}\right] + \frac{r^2+a^2}{\ell^2}\,,$$

$$h(\theta) = 1 + 2mA\cos\theta + \left[A^2(a^2+e^2) - \frac{a^2}{\ell^2}\right]\cos^2\theta\,,$$

$$\Sigma = 1 + \frac{a^2}{r^2}\cos^2\theta\,, \qquad H = 1 + Ar\cos\theta\,.$$

Check M, T and S

Using the usual Euclidean method, find temperature:

$$T = \frac{f'_+}{4\pi\alpha} = \frac{1}{2\pi r_+^2 \alpha}\left[m(1 - A^2 r_+^2) + \frac{r_+^3}{\ell^2(1 - A^2 r_+^2)}\right]$$

Which depends on alpha, and entropy:

$$S = \frac{\pi r_+^2}{K(1 - A^2 r_+^2)}$$

What is M?

Expand the metric near the boundary (Fefferman-Graham):

$$\frac{1}{r} = -A\xi - \sum F_n(\xi)\, z^n$$

$$\cos\theta = \xi + \sum G_n(\xi)\, z^n$$

F_n and G_n determined by the requirement that

$$ds^2 = -\ell^2 dz^2 + \frac{1}{z^2}\left[\gamma_{\mu\nu} + z^2\Psi_{\mu\nu} + z^3 M_{\mu\nu}\right] dx^\mu dx^\nu + \mathcal{O}(z^2)$$

Fefferman-Graham

For the boundary metric, get:

$$\frac{\left(1-A^2\ell^2 g(\xi)\right)^3}{\alpha^2\ell^2 F_1^2(\xi)}d\tau^2 - \frac{\left(1-A^2\ell^2 g(\xi)\right)}{F_1^2(\xi)g(\xi)}d\xi^2 - \frac{g(\xi)\left(1-A^2\ell^2 g(\xi)\right)^2}{K^2 F_1^2(\xi)}d\phi^2$$

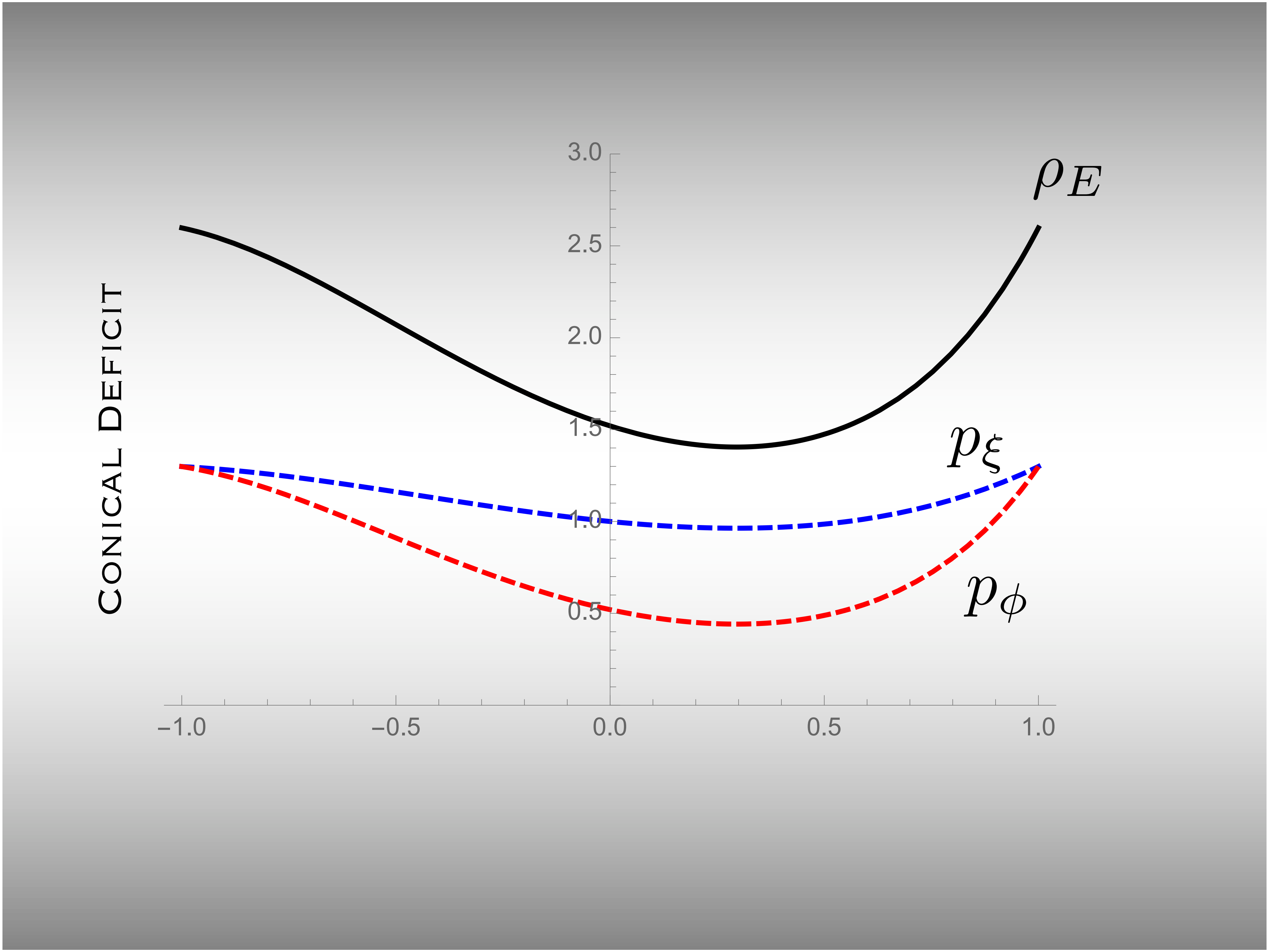
And for the boundary fluid stress tensor:

$$\langle \mathcal{T}_\nu^\mu \rangle = \mathrm{diag}\left\{\rho_E, -\rho_E/2+\Pi, \rho_E/2-\Pi\right\}$$

where

$$\rho_E = \frac{m}{\alpha}(1-A^2\ell^2 g)^{3/2}(2-3A^2\ell^2 g)$$

$$\Pi = \frac{3A^2\ell^2 g m}{2\alpha}(1-A^2\ell^2 g)^{3/2}$$



Conical Deficit
3.0
2.5
2.0
1.5
1.0
0.5
−1.0
−0.5
0.0
0.5
1.0
ρ_E
p_ξ
p_ϕ

Accelerating Thermodynamics

Integrate up the boundary stress-energy to get the mass:

$$M = \int \rho_E \sqrt{\gamma} = \frac{\alpha m}{K}$$

What is alpha? Setting m to zero, and demanding that the boundary is a round 2-sphere gives

$$\alpha = \sqrt{1 - A^2 \ell^2}$$

Get a consistent first law with corrections to V and TD length, and – can generalise to rotation

General Thermo Parameters

$$M = \frac{m(\Xi + a^2/\ell^2)(1 - A^2\ell^2\Xi)}{K\Xi\alpha(1 + a^2A^2)}$$

$$T = \frac{f'_+ r_+^2}{4\pi\alpha(r_+^2 + a^2)}\,, \quad S = \frac{\pi(r_+^2 + a^2)}{K(1 - A^2 r_+^2)}\,,$$

$$Q = \frac{e}{K}\,, \quad \Phi = \Phi_t = \frac{e r_+}{(r_+^2 + a^2)\alpha}\,,$$

$$J = \frac{ma}{K^2}\,, \quad \Omega = \Omega_H - \Omega_\infty\,, \quad \Omega_H = \frac{Ka}{\alpha(r_+^2 + a^2)}$$

$$P = \frac{3}{8\pi\ell^2}\,, \quad V = \frac{4\pi}{3K\alpha}\left[\frac{r_+(r_+^2 + a^2)}{(1 - A^2 r_+^2)} + \frac{m[a^2(1 - A^2\ell^2\Xi) + A^2\ell^4\Xi(\Xi + a^2/\ell^2)]}{(1 + a^2A^2)\Xi}\right]$$

$$\lambda_\pm = \frac{r_+}{\alpha(1 \pm A r_+)} - \frac{m}{\alpha}\frac{[\Xi + a^2/\ell^2 + \frac{a^2}{\ell^2}(1 - A^2\ell^2\Xi)]}{(1 + a^2A^2)\Xi^2} \mp \frac{A\ell^2(\Xi + a^2/\ell^2)}{\alpha(1 + a^2A^2)}$$

$$\Xi = 1 - \frac{a^2}{l^2} + A^2(e^2 + a^2) \qquad \alpha = \frac{\sqrt{(\Xi + a^2/\ell^2)(1 - A^2\ell^2\Xi)}}{1 + a^2A^2}$$

Chemical Expressions

$$M^2 = \frac{\Delta S}{4\pi}\left[\left(1+\frac{\pi Q^2}{\Delta S}+\frac{8PS}{3\Delta}\right)^2+\left(1+\frac{8PS}{3\Delta}\right)\left(\frac{4\pi^2 J^2}{(\Delta S)^2}-\frac{3C^2\Delta}{2PS}\right)\right]$$

$$V = \frac{2S^2}{3\pi M}\left[\left(1+\frac{\pi Q^2}{\Delta S}+\frac{8PS}{3\Delta}\right)+\frac{2\pi^2 J^2}{(\Delta S)^2}+\frac{9C^2\Delta^2}{32P^2S^2}\right],$$

$$T = \frac{\Delta}{8\pi M}\left[\left(1+\frac{\pi Q^2}{\Delta S}+\frac{8PS}{3\Delta}\right)\left(1-\frac{\pi Q^2}{\Delta S}+\frac{8PS}{\Delta}\right)-\frac{4\pi^2 J^2}{(\Delta S)^2}-4C^2\right],$$

$$\Omega = \frac{\pi J}{SM\Delta}\left(1+\frac{8PS}{3\Delta}\right),$$

$$\Phi = \frac{Q}{2M}\left(1+\frac{\pi Q^2}{S\Delta}+\frac{8PS}{3\Delta}\right),$$

$$\lambda_\pm = \frac{S}{4\pi M}\left[\left(\frac{8PS}{3\Delta}+\frac{\pi Q^2}{\Delta S}\right)^2+\frac{4\pi^2 J^2}{(\Delta S)^2}\left(1+\frac{16PS}{3\Delta}\right)-(1\mp 2C)^2\pm\frac{3C\Delta}{2PS}\right]$$

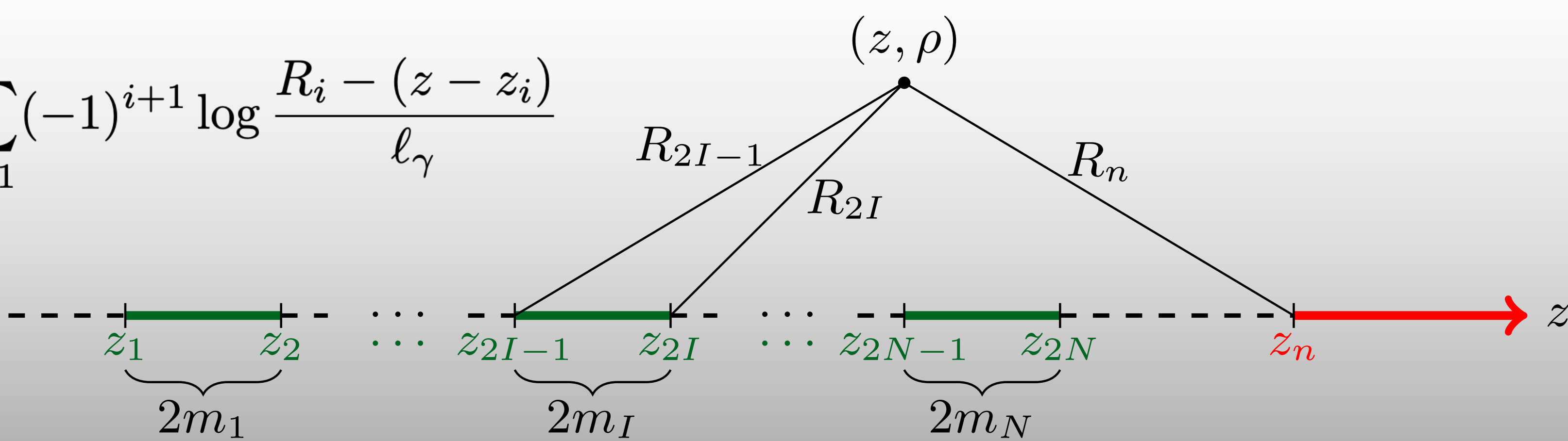
RG & Scoins, 1904.09660

Many Horizons?

What if we have multiple horizons? In vacuo, we can have multiple black holes along an axis separated by strings or struts: Bach-Weyl or Israel-Khan solutions.

$$ds^2 = e^{2\gamma} dt^2 - e^{2(\nu-\gamma)}(dr^2 + dz^2) - r^2 e^{-2\gamma} \frac{d\phi^2}{K^2}$$

$$2\gamma \sum_{i=1}^{n} (-1)^{i+1} \log \frac{R_i - (z - z_i)}{\ell_\gamma}$$



Dowker & Thambyapillai,CQG 20, 127 (2003) gr-qc/0105044

Thermodynamics

Yet again, with some fairly hefty algebraic manipulation, we were able to show that the combined black hole system satisfies a first law:

$$dM = \sum \left[T_I dS_I - \lambda_I d\mu_I \right]$$

With the thermodynamic length given by the worldsheet volume of the string connecting black holes.

RG, Lim, Scoins, Front. Phys. 8 187 (2021) arXiv:2012.15561

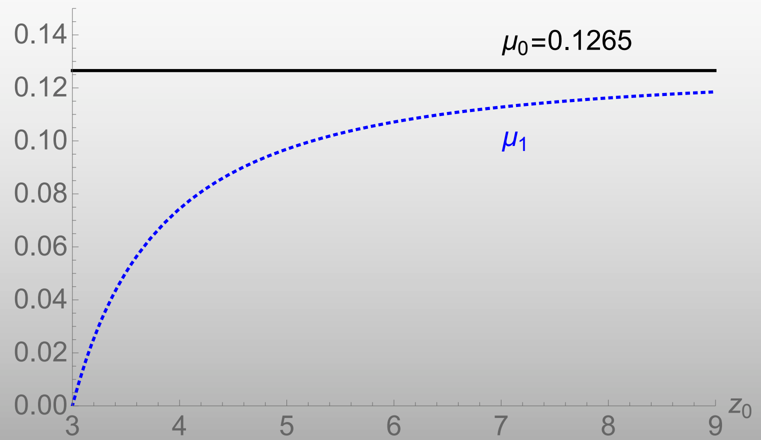
Three black holes:

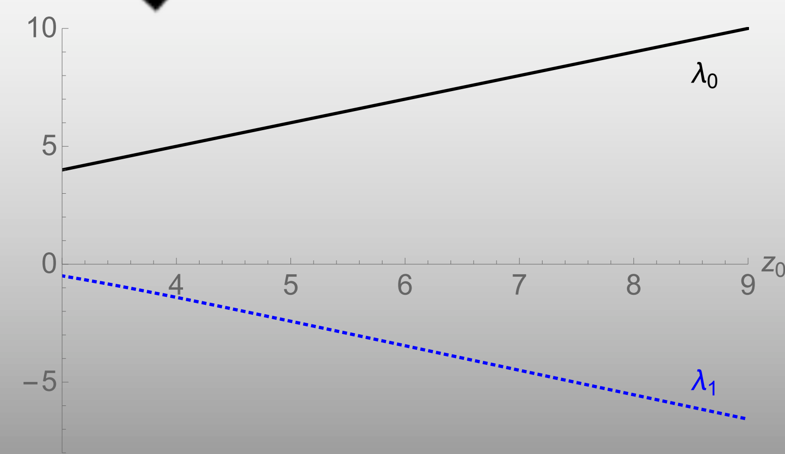
$$S_1 = \frac{4\pi m_0^2}{K}\frac{(z_0+m_0)(z_0+m_0+m)}{z_0(z_0+m_0-m)} = S_3$$

$$S_2 = \frac{4\pi m^2}{K}\frac{(z_0^2-m_0^2)(z_0+m_0+m)^2}{z_0^2(z_0-m_0+m)^2}$$

$$\mu_0 = \frac{1}{4}\left(1-\frac{1}{K}\right)$$

$$\mu_1 = \frac{1}{4}\left(1-\frac{z_0^2(z_0^2-(m_0-m)^2)}{(z_0^2-m_0^2)(z_0^2-(m_0+m)^2)K}\right) = \mu_2$$





Quick Recap

Now can derive thermodynamics of complicated systems of multiple black holes connected by possibly infinite strings.

The charged rotating slowly accelerating C-metric has 6 parameters:

$$m \quad \ell \quad Q \quad a \quad A \quad K$$

and we have six charges:

$$M \quad P \quad Q \quad J \quad \mu_+ \quad \mu_-$$

<u>Full cohomogeneity</u>

Generally, claims of bizarre thermodynamics related to non-full cohomogeneity.

Unique First Law?

To some extent, finding a first law by consistency is unsatisfying! There are claims of alternate first laws, but these tend to have constrained parameters.

While we do not have a proof, the existence of the chemical expressions and the full cohomogeneity are compelling reasons to conclude we have the correct First Law.

What is dM?

The original discussion of Bardeen, Carter, and Hawking gave a clear relation for dM, the change in mass of the black hole, in terms of infalling matter. For a vacuum spacetime, this is unambiguous as we can measure mass at infinity.

For de Sitter however, we can use the slow-roll solution to explore what is dM. The scalar field slowly rolls and accretes onto the black hole.

Scalar Field Eqn

Idea is to turn e.o.m for ϕ

$$\frac{\phi_{,tt}}{f} - \frac{1}{r^2}\left(r^2 f \phi_{,r}\right)_{,r} = -\frac{\partial W}{\partial \phi}$$

into something like a slow roll equation by assuming $\phi = \phi(T)$, where

$$T = t + \xi(r)$$

T is constructed so that ϕ is regular at both horizons, with only in(out) going modes at black hole (cosmological) horizon.

Substitute in: $\phi = \phi(t + \xi(r))$

$$\frac{1}{r^2}\left(r^2 f \xi'\right)' \dot{\phi} - \frac{\ddot{\phi}}{f}\left(1 - f^2 \xi'^2\right) = \frac{\partial W}{\partial \phi}$$

Dropping second term, and remember $\phi = \phi(T)$, we must have

$$\frac{1}{r^2}\left(r^2 f \xi'\right)' = -3\gamma$$

γ constant, and hence

$$\xi' = \frac{1}{f}\left(-\gamma r + \frac{\beta}{r^2}\right)$$

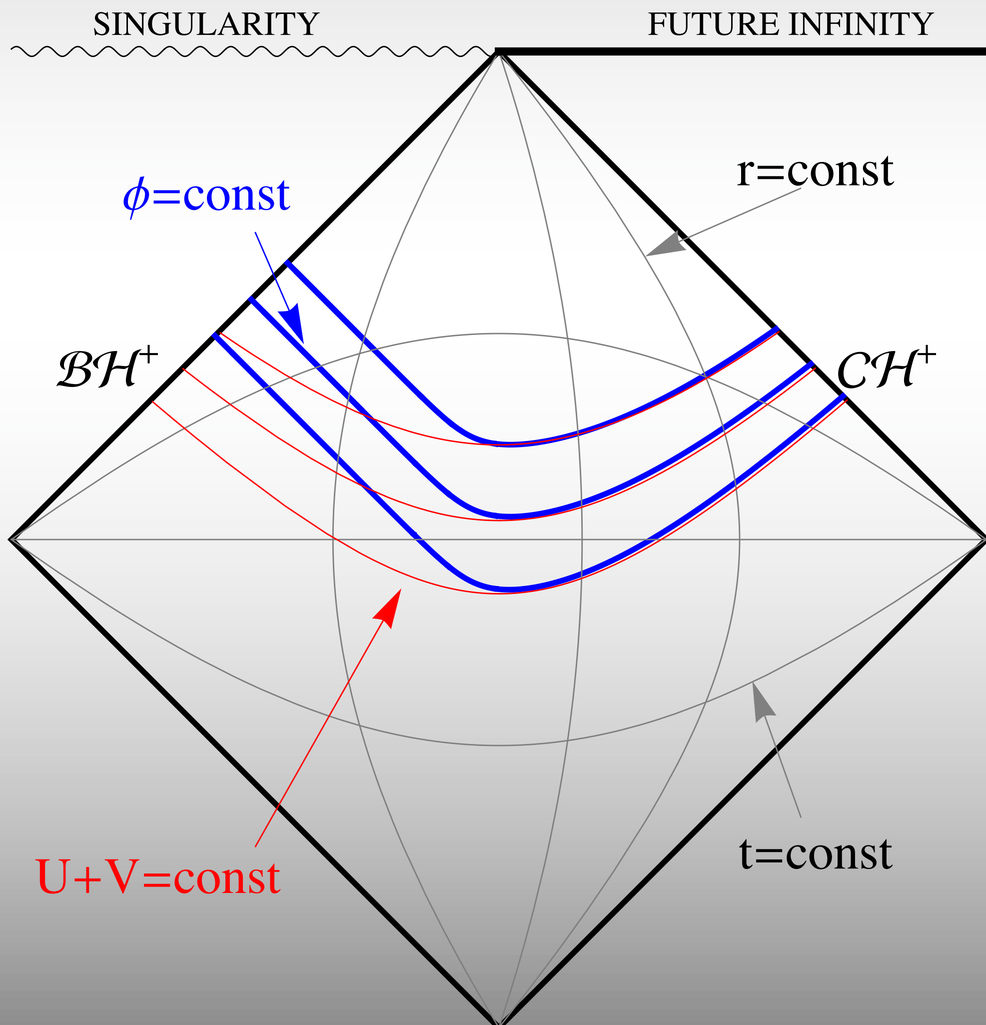
Find γ and β by regularity: $\phi(T)$ must be ingoing on event horizon and outgoing on cosmological horizon.
Final answer gives T:

$$T = t - \frac{1}{2\kappa_c} \log \left| \frac{r - r_c}{r_c} \right| + \frac{1}{2\kappa_b} \log \left| \frac{r - r_b}{r_b} \right| + \frac{r_b r_c}{r_c - r_b} \log \frac{r}{r_0} + \left(\frac{r_c}{4\kappa_b r_b} - \frac{r_b}{4\kappa_c r_c} \right) \log \left| \frac{r - r_n}{r_n} \right|$$

T looks like Kruskal V at the black hole horizon (r_b) and Kruskal U at the cosmological horizon (r_c)

The T Coordinate

The T coordinate is timelike at each horizon, and could be a cosmological time asymptotically.



Accretion

We can solve the Einstein equations perturbatively in the slow roll parameters, and obtain expressions for the change in M and Λ

$$\dot{M}(T) = 4\pi\beta\dot{\phi}^2 \qquad \dot{\Lambda}(T) = -3\gamma\frac{\dot{\phi}^2}{M_p^2}$$

On the horizon: $\dot{M}(T) = 4\pi\beta T^r_V$

Where

$$\beta = \frac{r_b^2 r_c^2(r_b + r_c)}{r_c^3 - r_b^3} \geq r_b^2$$

i.e. the increase in mass is not the energy-momentum crossing the horizon. Instead:

$$4\pi r_b^2 T^r_T = T\dot{S}$$

Summary

- Have shown how to include composite systems in black hole thermodynamics
- Conjugate variable for tension is *Thermodynamic Length*
- Thermodynamics of accelerating black holes is computable – non-static and non-isolated.
- We can also do time dependent thermodynamics and construct more nuanced black hole solutions.
- How much do we trust these techniques?