Positivity Bounds and Effective Field Theories
A brief review

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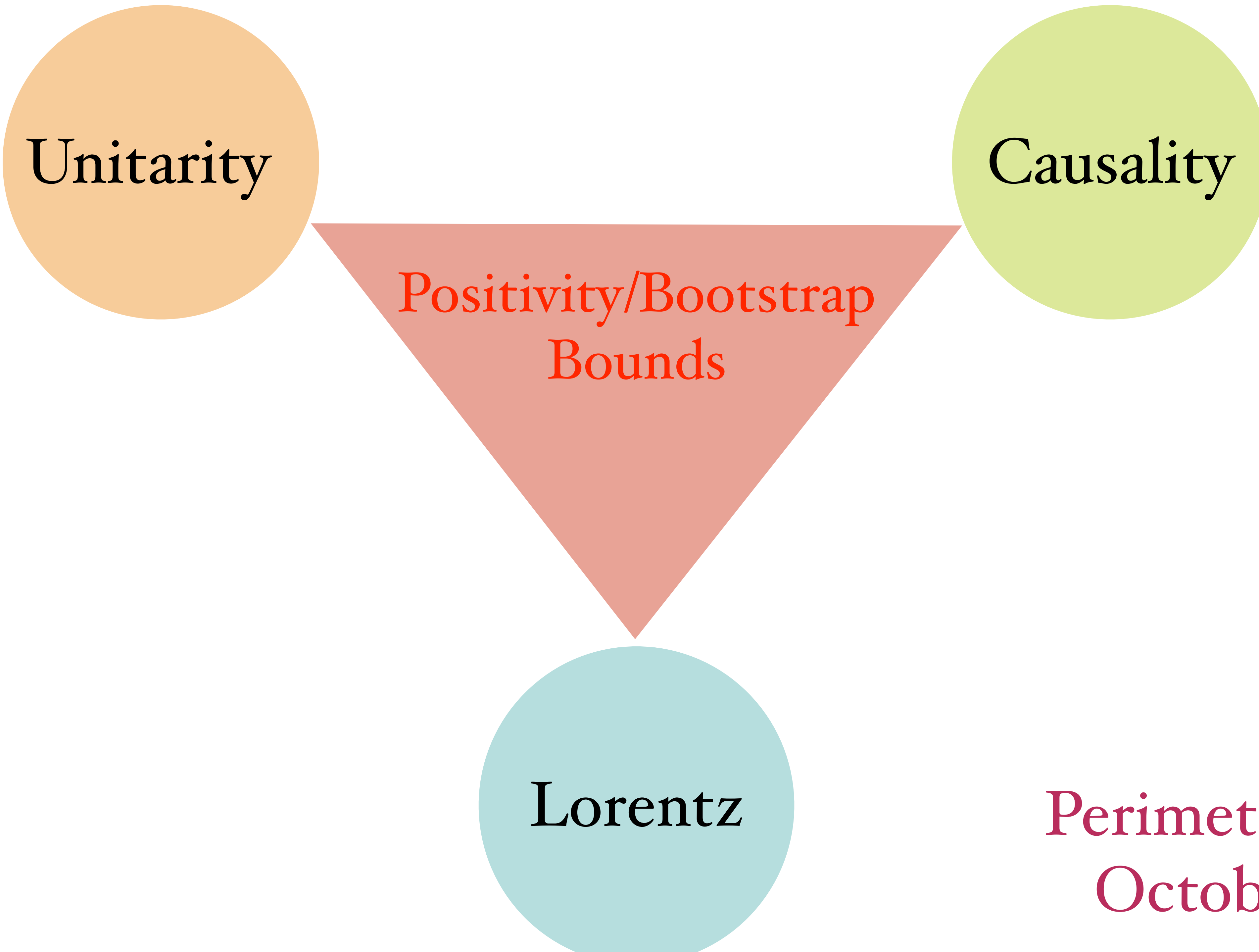


Perimeter Institute,
October 23 2023

Are all EFTs allowed?

With typical assumption that:
UV completion is Local, Causal, Poincare Invariant and Unitary

Answer: NO! Certain low energy effective theories do not admit well defined UV completions

Positivity Bounds/S-matrix Bootstrap

- Place constraints on signs and magnitudes of irrelevant operators in an EFT - Particular fruitful for EFTs of higher spin particles and EFTs in broken states (i.e. for Goldstone/ Stuckelberg modes)
- Most constraints are double sided (compact bounds!)
- Bounds broadly consistent with naturalness/EFT power counting arguments

Non-relativistic Causality/Analyticity

Causal propagation:

$$G_{\rm ret}(t,t') = \theta(t-t')\Delta(t,t')$$

In momentum space:

$$G_{\rm ret}(\omega) = \int_{-\infty}^{\infty} dt\, e^{i\omega(t-t')} G_{\rm ret}(t,t')$$

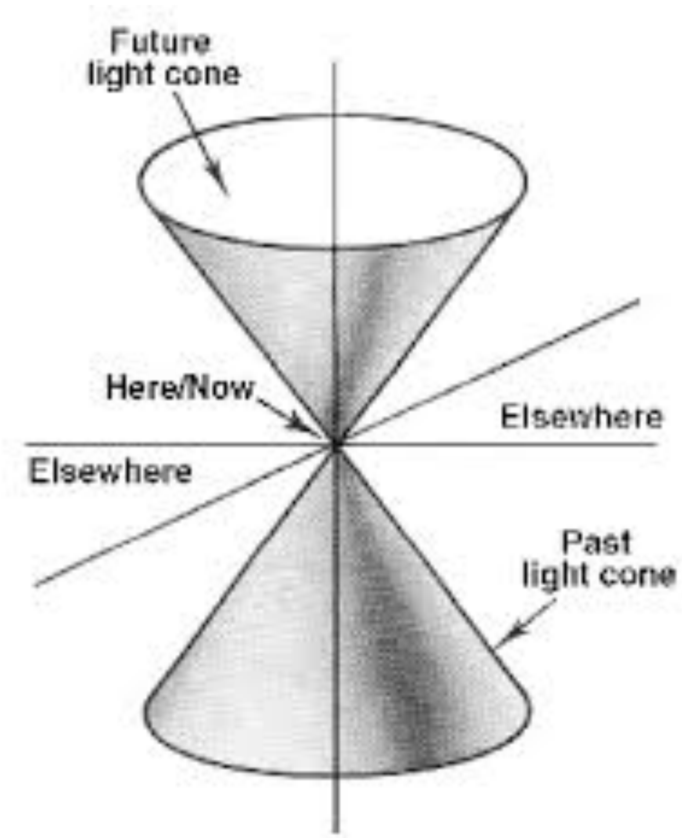
$$= \int_{0}^{\infty} dt\, e^{i\omega t} \Delta(t,0)$$

Analytic in upper-half complex plane

Causality implies analyticity!!!!

Lets add relativity!

Suppose we have a scalar operator $\hat{O}(x)$



Relativistic Locality tells us that ......

$$[\hat{O}(x), \hat{O}(y)] = 0$$

$$\text{if } (x-y)^2 > 0$$

Unitarity (positivity) tells us that

$\langle\psi|\hat{O}(f)^2|\psi\rangle > 0$ where $\hat{O}(f) = \int d^4x\, f(x)\hat{O}(x)$

Kallen-Lehmann Spectral Representation

Together with Poincare invariance these imply:

$$i\langle 0|\hat{T}\hat{O}(x)\hat{O}(y)|0|\rangle = \int \frac{d^d k}{(2\pi)^d} e^{ik.(x-y)} G_O(k)$$

$$G_O(k) = \frac{Z}{k^2+m^2-i\epsilon} + S(-k^2) + (-k^2)^N \int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(k^2+\mu-i\epsilon)}$$

$$S(-k^2) = \sum_{k=0}^{N-1} c_k(-k^2)^k \qquad \lim_{\mu\to\infty} \rho(\mu) \sim \mu^{\Delta-d/2} \qquad N = [\Delta - d/2 + 1]$$

Δ UV Conformal weight

Positive Spectral Density
as a result of Unitarity

$$\rho(\mu) \geq 0$$

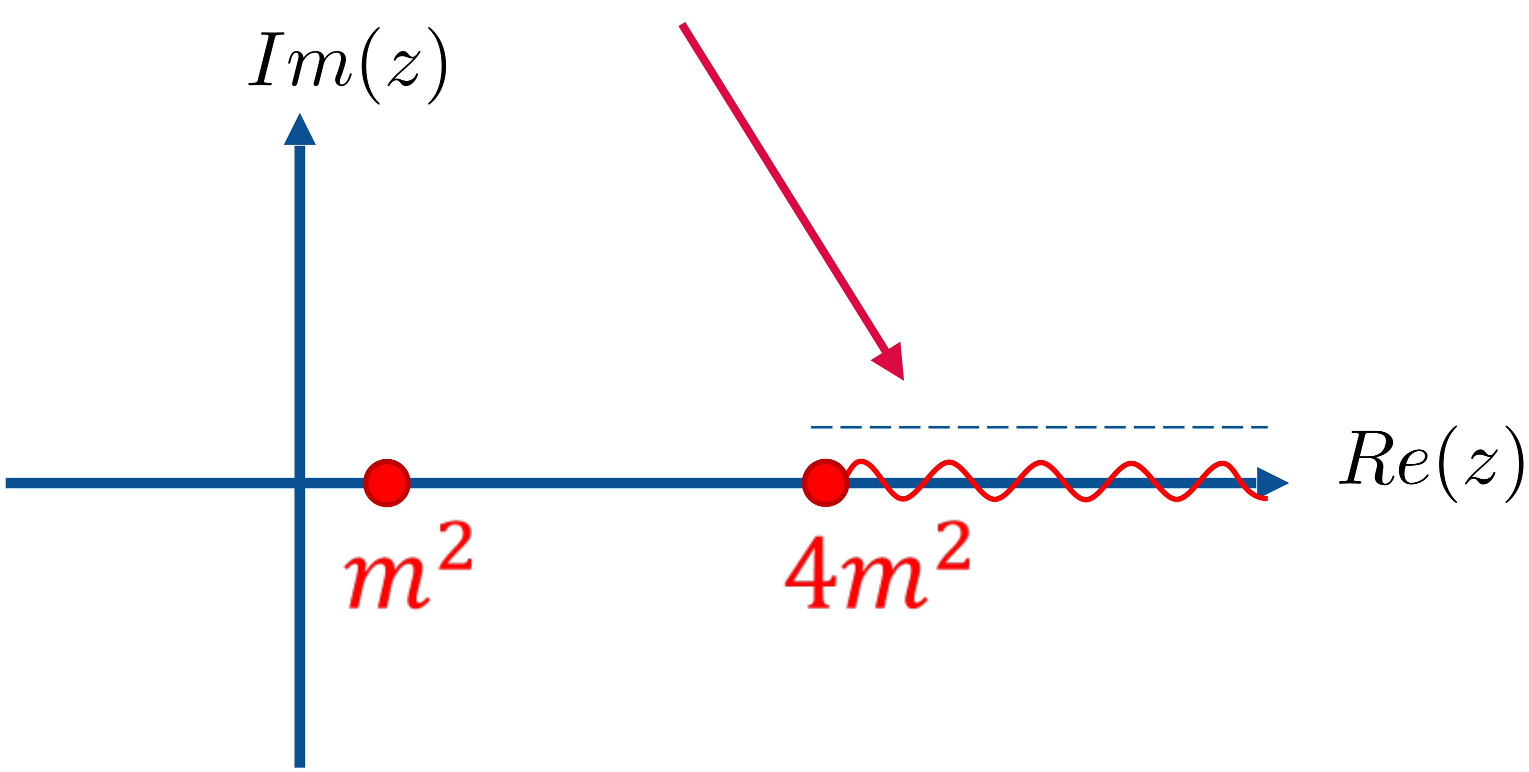
Analytic Structure

Define complex momenta squared $\quad z = -k^2 + i\epsilon$

Pole $\qquad$ Branch cut

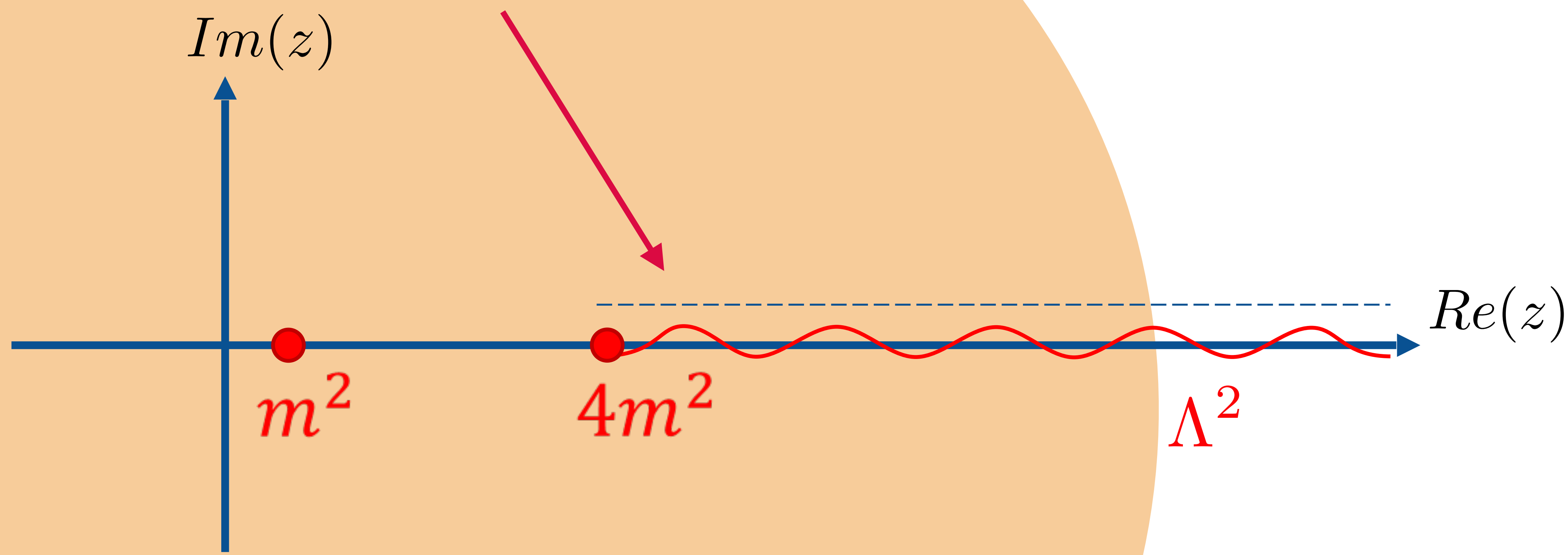
$$G_O(z) = \boxed{\frac{Z}{m^2 - z}} + S(z) + z^N \boxed{\int_{4m^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}}$$

Physical region



Region of Validity of EFT

Physical region

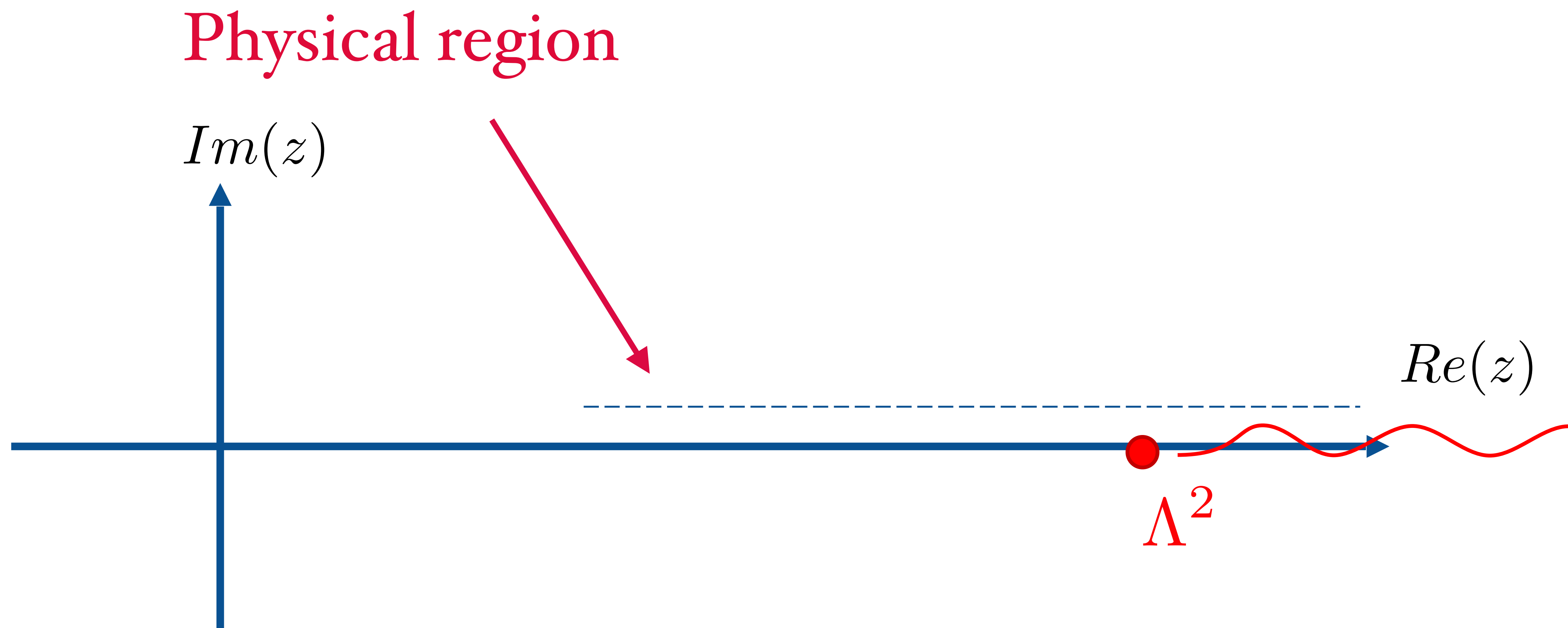


LEEFT valid here, can calculate pole and LE part of cut

UV completion - unknown?

Analytic Structure 2: Move the branch cut!

de Rham, Melville, AJT, Zhou 1702.08577
Bellazzini et al 1710.02539
de Rham, Melville, AJT 1710.09611



$$G'_O(z) = G_O(z) - \frac{Z}{m^2 - z} - z^N \int_{4m^2}^{\Lambda^2} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$

$$G'_O(z) = S(z) + z^N \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$

Calculable in EFT

Linear Positivity Bounds

$$G'_O(z) = S(z) + z^N \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^N(\mu - z)}$$

$M \geq N$

$$D_M(z) = \frac{1}{M!}\frac{d^M}{dz^M} G'_O(z) = \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{(\mu - z)^{M+1}}$$

$$D_M(0) = \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{M+1}}$$

$$D_M(0) > 0 \qquad\qquad D_M(0) \geq \Lambda^2 D_{M+1}(0)$$

Positivity of these integrals enforces positivity of combinations of Wilson coefficients for Irrelevant operators

Nonlinear Moment Positivity

Maths by Stieltjes in 1890s, applied to scattering amplitudes positivity in 1970s!! Rejuvenated in: Arkani-Hamed, Huang, Huang EFT-Hedron 2020
Bellazzini et al, Positive Moments .., 2020

$$\boxed{D_M(0) = \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{M+1}} = \langle \frac{1}{\mu^M} \rangle}$$

$$y^T D_M y = \sum_{p,q=0}^{N} D_{M+p+q} y^p y^q = \langle \mu^{-M} (\sum_{p=0}^{N} y^p \mu^{-p})^2 \rangle > 0$$

$$\downarrow$$

'positivity of N x N Hankel matrix'

$$det(D_M) > 0$$

$$(D_M)_{pq} = D_{M+p+q}$$

Nonlinear Moment Positivity

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$$D_M(0) = \int_{\Lambda^2}^{\infty} d\mu \frac{\rho(\mu)}{\mu^{M+1}} = \langle \frac{1}{\mu^M} \rangle$$

Simply example **Cauchy-Schwarz:**

$$\langle (\mu^{-M} + \lambda \mu^{-N})^2 \rangle \geq 0$$

$$D_{2M} D_{2N} \geq (D_{N+M})^2$$

$$\begin{pmatrix} D_{2N} & D_{N+M} \\ D_{N+M} & D_{2M} \end{pmatrix}$$

'positivity of 2 x 2 Hankel matrix'

Repeated use of Cauchy Schwarz:

$$D_{2N} \geq \frac{D_1^{2N}}{D_0^{2N-1}}$$

What does this tell us about EFT?

e.g. Suppose scalar field in EFT with tree level action ....

$$S = \int d^4x \hat{O}(x)[\Box + a_1 \frac{\Box^2}{\Lambda^2} + a_2 \frac{\Box^3}{\Lambda^4} + \ldots]\hat{O}(x)$$

Tree level Feynman propagator is

$$G_O(z) = -\frac{1}{z + a_1 \frac{z^2}{\Lambda^2} + a_2 \frac{z^3}{\Lambda^4} + a_3 \frac{z^4}{\Lambda^6} + a_4 \frac{z^5}{\Lambda^8} \ldots}$$

$$\downarrow$$

$$G'_O(z) = \frac{a_1}{\Lambda^2} + \frac{(a_2 - a_1^2)}{\Lambda^4} z + \frac{a_1^3 - 2a_1 a_2 + a_3}{\Lambda^6} z^2 + \frac{a_4 - 2a_1 a_3 - a_2^2 + 3a_1^2 a_2 - a_1^4}{\Lambda^8} z^3 + \mathcal{O}(z^4)$$

What does this tell us about EFT?

$$G'_O(z) = \frac{a_1}{\Lambda^2} + \frac{(a_2 - a_1^2)}{\Lambda^4} z + \frac{a_1^3 - 2a_1 a_2 + a_3}{\Lambda^6} z^2 + \frac{a_4 - 2a_1 a_3 - a_2^2 + 3a_1^2 a_2 - a_1^4}{\Lambda^8} z^3 + \mathcal{O}(z^4)$$

assuming no subtractions $N = 0$

Linear Positivity Bounds:

$$a_1 > 0 \qquad a_2 > a_1^2$$
$$a_1^3 - 2a_1 a_2 + a_3 > 0$$
$$a_4 - 2a_1 a_3 - a_2^2 + 3a_1^2 a_2 - a_1^4 > 0$$

NonLinear Positivity Bounds:

$$D_2 D_0 > D_1^2 \longrightarrow a_1 a_3 - a_2^2 > 0$$

$$D_3 D_0^2 - D_1^3 + 2D_0^2(D_2 D_0 - D_1^2) > 0 \longrightarrow a_4 a_1^2 - a_2^3 > 0$$

All particles are elementary, but some are more elementary than others. (Abdus Salam 1960)

S-Matrix lore

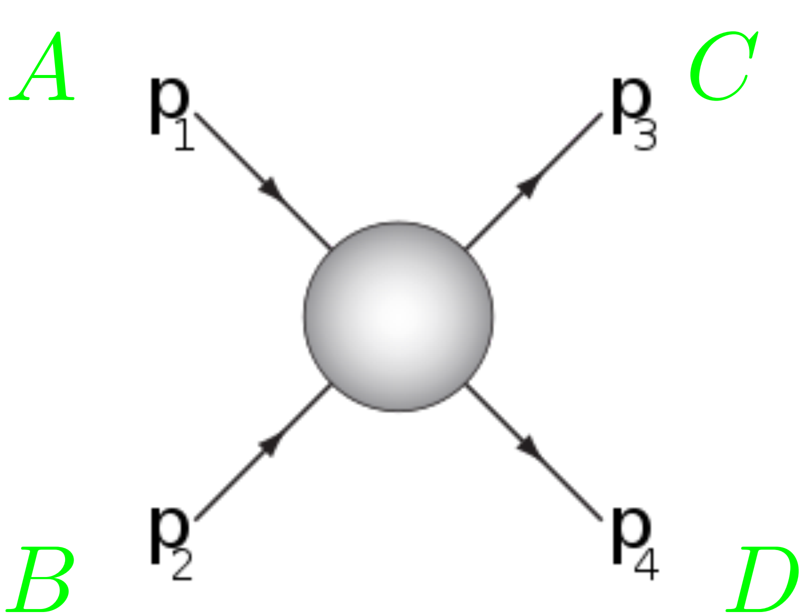
$$|A(k)| < \alpha e^{\beta|k|}$$

1. Unitarity $\quad S^\dagger S = 1$
2. Locality: Scattering Amplitude Polynomially (Exponentially) Bounded
3. Causality: Analytic Function of Mandelstam variables (modulo poles+cuts)
4. Poincare Invariance
5. Crossing Symmetry: Follows from above assumptions
6. Mass Gap: Existence of Mandelstam Triangle and Validity of Froissart Bound

Added Ingredient: Crossing Symmetry

s-channel

$$A + B \to C + D$$



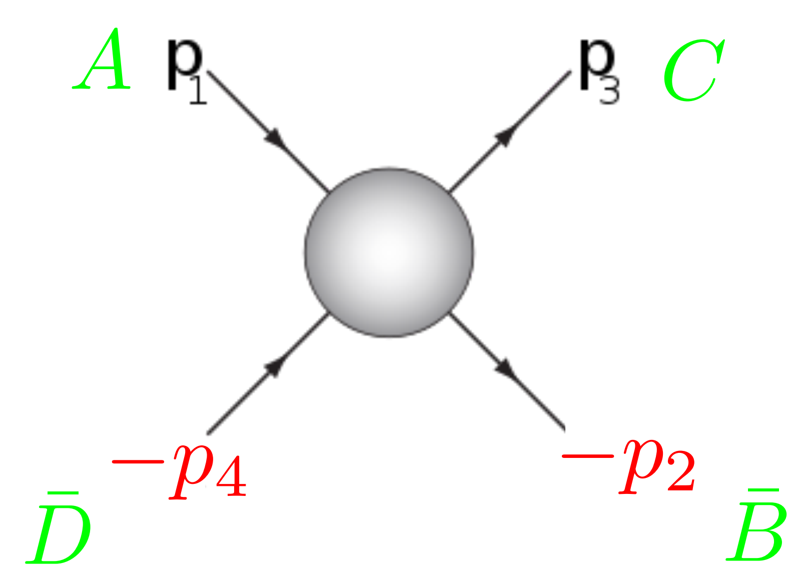
$$s = -(p_1 + p_2)^2$$

$$t = -(p_1 + p_3)^2$$

$$u = -(p_1 + p_4)^2$$

u-channel

$$A + \bar{D} \to C + \bar{B}$$



S-matrix analyticity

Punch Line:

In a theory with a mass-gap, the 2-2 Scattering amplitude $A(s,t)$ is an analytic function of s at fixed $t < 4m^2$ with poles and branch cuts in physical places.

$\partial_t^n Im[A(s,t)] > 0$ Unitarity of partial waves $Im[a_l(s)] > 0$

Jin and Martin bound (1964) $|A(s,t)| < s^2 \qquad 0 \leq t < 4m^2$

We can write a dispersion relation with two subtractions!

1970's Positivity Constraints

Positivity bounds first developed around 1970 - many different statements and different methods - including ones that emphasise use of full crossing symmetry

Positivity refers to restricted requirement

$$\mathrm{Im}(a_l(s)) \geq 0 \qquad s \geq 4m^2$$

as opposed to full unitarity!

(as used for example in S-matrix bootstrap program)

$$0 \leq |a_l(s)|^2 \leq \mathrm{Im} a_l(s) \leq 1 \qquad s \geq 4m^2$$

Focus mainly on bounds on partial waves $a_l(s)$ in 'unphysical' region $0 \leq s < 4m^2$

Partial wave expansion

Partial waves can be inferred from amplitude by orthogonality of Legendre polynomials

$$A(s,t) = 16\pi\sqrt{\frac{s}{s-4m^2}}\sum_{l=0}^{\infty}(2l+1)P_l\left(1+\frac{2t}{s-4m^2}\right)a_l(s)$$

$$a_l(s) = \frac{1}{16\pi}\sqrt{\frac{s-4m^2}{s}}f_l(s) \qquad z = \cos\theta = 1 + 2t/(s-4m^2)$$

$$\boxed{f_l(s) = \frac{1}{2}\int_{-1}^{1} dz\, P_l(z)A(s,z)}$$

Froissart-Gribov representation

$$f_l(s) = \frac{1}{2}\int_{-1}^{1} dz\, P_l(z) A(s,z)$$

Using $Q_l(z) = \frac{1}{2}\int_{-1}^{1} dz' \frac{P_l(z')}{z-z'}$ and dispersion relation

$$f_l(s) = \frac{4}{\pi(4m^2-s)}\int_0^\infty d\mu\, Q_l\left(-1+\frac{2\mu}{4m^2-s}\right) \mathrm{Im} A(\mu,s)$$

$l = 2, 4, 6 \ldots$ Odd partial waves vanish by t-u crossing symmetry

$l = 0$ determined by subtraction function so no obvious positivity

Froissart-Gribov representation

Using integral representation:

$$Q_l(z) = \int_0^\infty d\theta \left(z + (z^2 - 1)\cosh\theta\right)^{-l-1}$$

One can prove:

$$f_l(s) = \frac{4}{\pi(4m^2 - s)} \int_0^\infty d\mu \int_0^\infty d\theta (z(\mu) + (z(\mu)^2 - 1)\cosh\theta)^{-l-1} \mathrm{Im} A(\mu, s)$$

$$l = 2, 4, 6 \ldots \qquad z(\mu) = -1 + \frac{2\mu}{4m^2 - s}$$

Stieljes argument

$$z(\mu) = -1 + \frac{2\mu}{4m^2 - s}$$

$$\sum_{p=0}^{n}\sum_{q=0}^{n} f_{l+p+q} y^p y^q =$$

$$\frac{4}{\pi(4m^2 - s)} \int_0^\infty d\mu \int_0^\infty d\theta (z(\mu) + (z(\mu)^2 - 1)\cosh\theta)^{-l-1}$$

$$\left[\sum_{p=0}^{n} y^p \int_0^\infty d\theta (z(\mu) + (z(\mu)^2 - 1)\cosh\theta)^{-p}\right]^2 \mathrm{Im} A(\mu, s)$$

Positive for $s < 4m^2$

Positive for $s \geq 0$

Totally Positive for $0 \leq s < 4m^2$

Positivity of Hankel Determinant

$$\sum_{p=0}^{n}\sum_{q=0}^{n} f_{l+p+q}\, y^p y^q \geq 0 \qquad\qquad 0 \leq s < 4m^2$$

'positivity of N x N Hankel matrix'

$$(F_l)_{pq} = f_{l+p+q}$$

$$\mathrm{Det}(F_l(s)) \geq 0 \qquad\qquad 0 \leq s < 4m^2$$

Example $\qquad f_6(s) f_2(s) \geq f_4(s)^2$

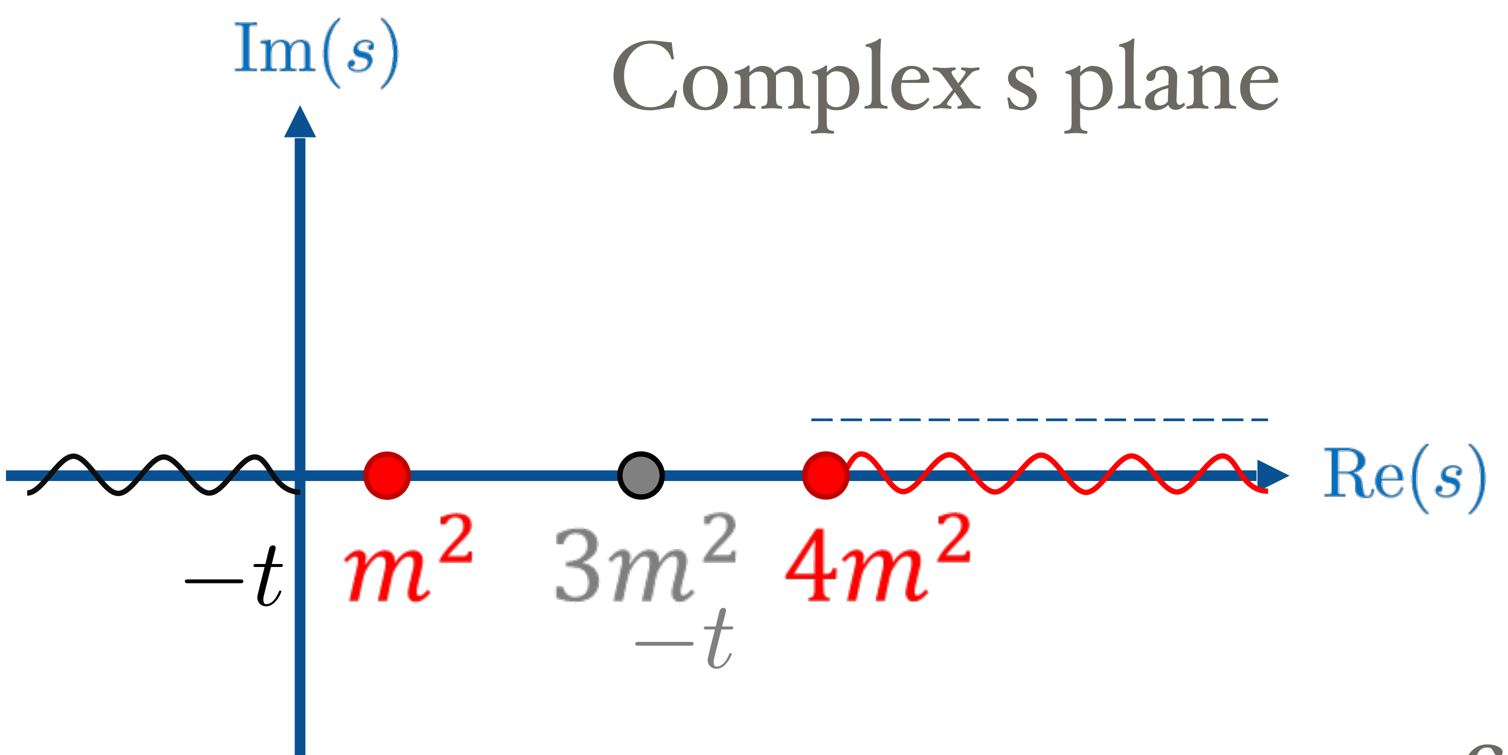
Infinite set of non-linear positivity constraints on partial waves amplitudes!!!!

2000-2020's Positivity Constraints

Key difference to 1970's

- Greater emphasis on constraints on Low energy effective theory Wilson coefficients rather than partial waves themselves
- Primarily interested in physical scattering region rather than Mandelstam triangle
- Application to theories without a mass gap (assuming weak coupling)

Scattering Amplitude Analyticity



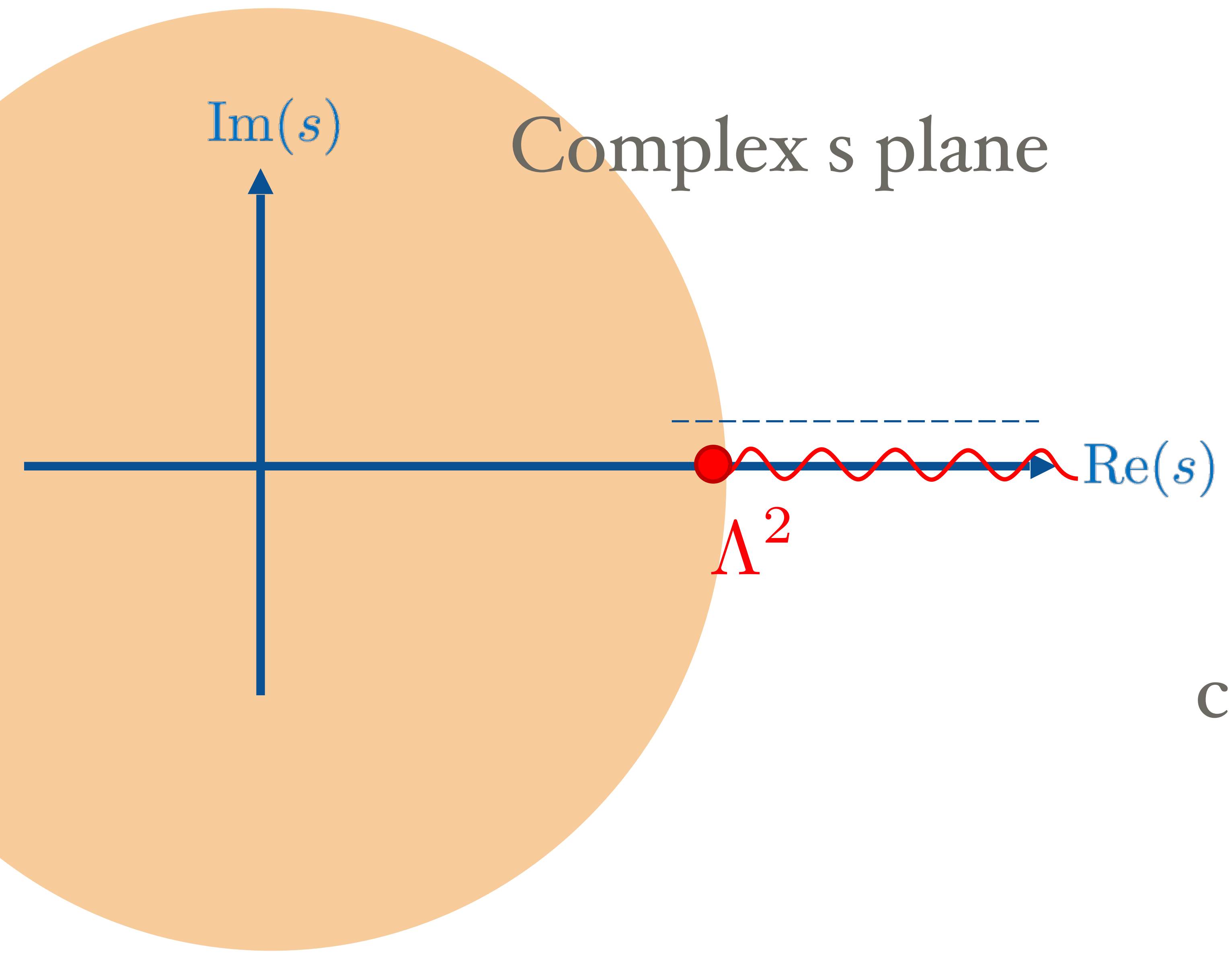
Physical scattering region is $s \geq 4m^2$

crossing: $u = 4m^2 - s - t$

$$\mathcal{A}_s(s,t) = \boxed{\frac{\lambda_s(t)}{m^2 - s} + \frac{\lambda_u(t)}{m^2 - u}} + \boxed{(c_0(t) + c_1(t)s)} + \boxed{\frac{s^2}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Im(A_s(\mu,t))}{\mu^2(\mu - s)} + \frac{u^2}{\pi}\int_{4m^2}^{\infty} d\mu \frac{Im(A_u(\mu,t))}{\mu^2(\mu - u)}}$$

Poles Subtractions Branch cuts

'Improved' Scattering Amplitude Analyticity



Physical scattering region is $s \geq 4m^2$

crossing: $u = 4m^2 - s - t$

$$\mathcal{A}'_s(s,t) = c_0(t) + c_1(t)s + \frac{s^2}{\pi}\int_{\Lambda^2}^{\infty} \frac{Im(A_s(\mu,t)}{\mu^2(\mu - s)} + \frac{u^2}{\pi}\int_{\Lambda^2}^{\infty} \frac{Im(A_s(\mu,t)}{\mu^2(\mu - u)}$$

Fixed t Stieltjes Positivity Bounds

$$\frac{1}{M!}\frac{d^M}{ds^M}\mathcal{A}'_s(2m^2-t/2,t)=\frac{1}{\pi}\int_{\Lambda^2}^{\infty}d\mu\frac{Im\mathcal{A}_s(\mu,t)+Im\mathcal{A}_s(\mu,t)}{(\mu-2m^2+t/2)^{M+1}}>0$$

$$0\le t<4m^2$$

$$\det_{pq}\left(\frac{1}{(M+p+q)!}\frac{d^{M+p+q}}{ds^{M+p+q}}\mathcal{A}'_s(2m^2-t/2,t)\right)>0$$

Even M+p+q

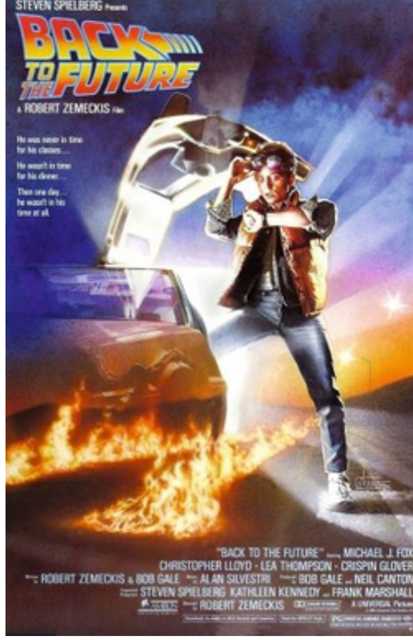
$M\ge 2$

Even M

$$0\le t<4m^2$$

Positivity of Goldstones and Pions

T. N. Pham and Tran N. Truong (1985)



Consider Goldstone EFT $\quad \mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{\Lambda^4}(\partial\phi)^4$

Positivity Bounds: $\quad \partial_s^2 A_s = \frac{c}{\Lambda^4} > 0$

Consider pion chiral Lagrangian

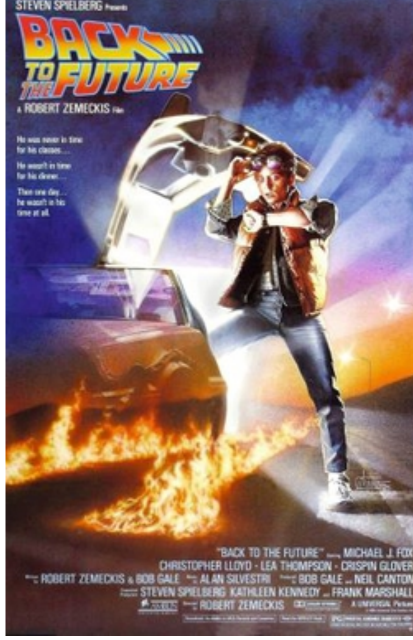
$$\mathcal{L}_Q = \frac{1}{32e^2}\mathrm{Tr}([\partial_\mu M M^\dagger, \partial_\nu M M^\dagger]^2) + \frac{\gamma}{8e^2}[\mathrm{Tr}(\partial_\mu M\, \partial_\mu M^\dagger)]^2$$

$$T^Q_{+0}(s,t,u) = \left(\frac{1}{e^2 f_\pi{}^4}\right)[(s-2m_\pi{}^2)^2 + (u-2m_\pi{}^2)^2 - 2(t-2m_\pi{}^2)^2] + \frac{\gamma}{e^2}\frac{1}{f_\pi{}^4}(t-2m_\pi{}^2)^2 \,,$$

$$T^Q_{00}(s,t,u) = \frac{2\gamma}{e^2}\frac{1}{f_\pi{}^4}[(s-2m_\pi{}^2)^2 + (t-2m_\pi{}^2)^2 + (u-2m_\pi{}^2)^2] \,,$$

Positivity Bounds: $\quad \frac{1}{e^2} > 0 \qquad \gamma > 0$

Positivity Bounds = (Sub)luminality



Adams et. al. 2006

For Goldstone model:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 + \frac{c}{\Lambda^4}(\partial\phi)^4$$

Positivity requires: $c > \alpha c^2 > 0$

(Sub)luminality requires $c_s^2 = 1 - \frac{c}{\Lambda^4}\dot{\phi}^2 < 1$

Positivity of scattering time delay: $c > 0$

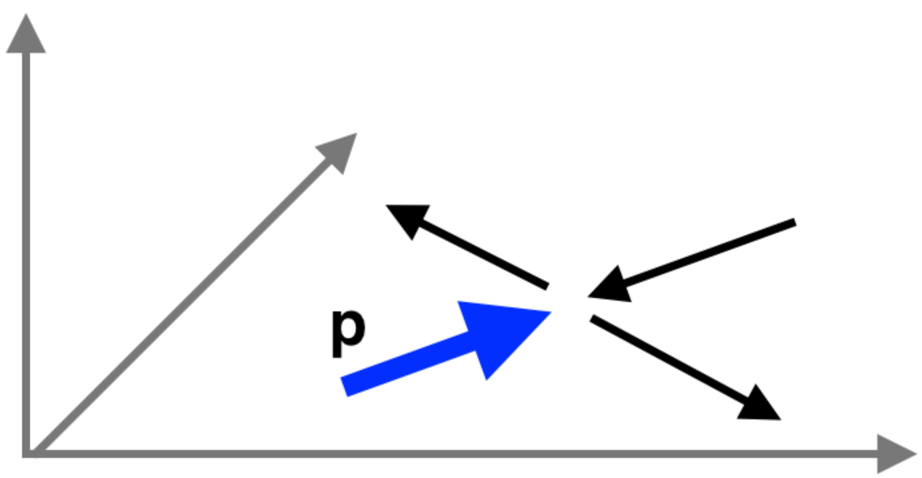
Makes sense since positivity derivation relies on Analyticity=Causality

de Rham, Melville, AJT, Zhou 1706.02712

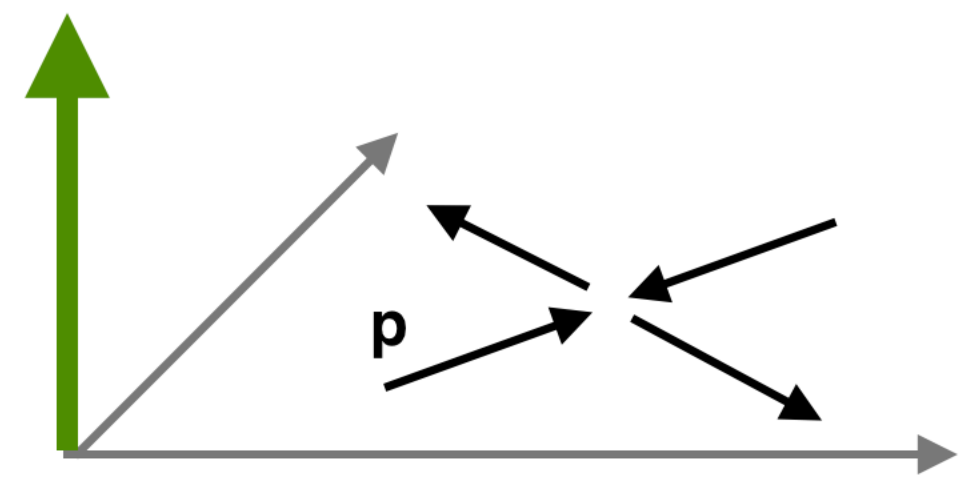
Scattering of all spins

Kotanski, 1965

Helicity



Transversity



$$\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4} = \sum_{\lambda_1\lambda_2\lambda_3\lambda_4} u^{S_1}_{\lambda_1\tau_1} u^{S_2}_{\lambda_2\tau_2} u^{S_1*}_{\tau_3\lambda_3} u^{S_2*}_{\tau_4\lambda_4} \mathcal{H}_{\lambda_1\lambda_2\lambda_3\lambda_4}$$

Change of Basis $u^S_{\lambda\tau} = \langle S, \lambda | e^{-i\frac{\pi}{2}\hat{J}_z} e^{-i\frac{\pi}{2}\hat{J}_y} e^{i\frac{\pi}{2}\hat{J}_z} | S, \tau\rangle$

$$T^s_{\tau_1\tau_2\tau_3\tau_4}(s,t,u) = e^{-i\sum_i \tau_i \chi} T^u_{-\tau_1-\tau_4-\tau_3-\tau_2}(u,t,s)$$

Crossing is Simple!!

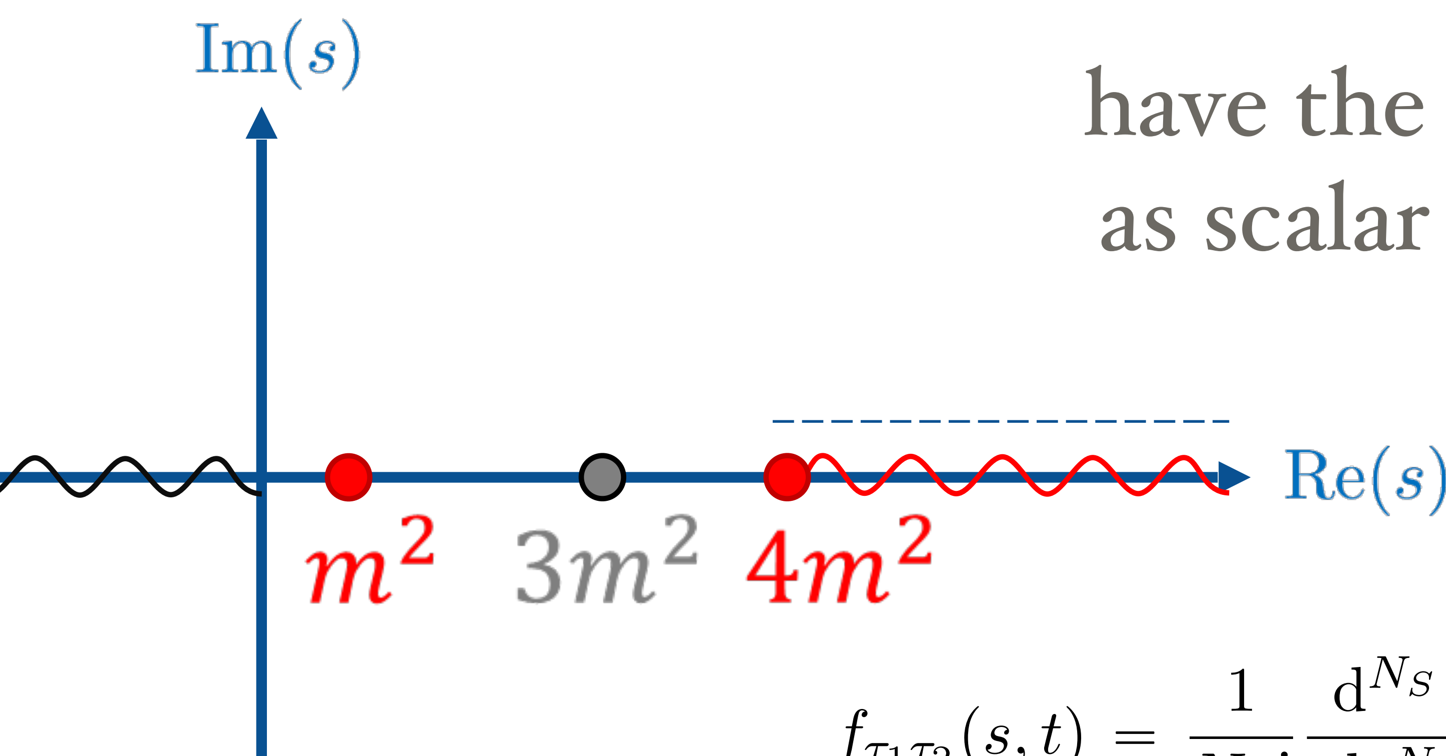
Dispersion Relation with Positivity along <u>BOTH</u> cuts

de Rham, Melville, AJT, Zhou 1706.02712

Punch line: The specific combinations:

$$\mathcal{T}^{+}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) = \left(\sqrt{-su}\right)^{\xi}\mathcal{S}^{S_1+S_2}\left(\mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,\theta) + \mathcal{T}_{\tau_1\tau_2\tau_3\tau_4}(s,-\theta)\right)$$

have the same analyticity structure as scalar scattering amplitudes!!!!!!!



Implies Dispersion Relation

$$f_{\tau_1\tau_2}(s,t) = \frac{1}{N_S!}\frac{\mathrm{d}^{N_S}}{\mathrm{d}s^{N_S}}\tilde{\mathcal{T}}^{+}_{\tau_1\tau_2\tau_1\tau_2}(s,t)$$

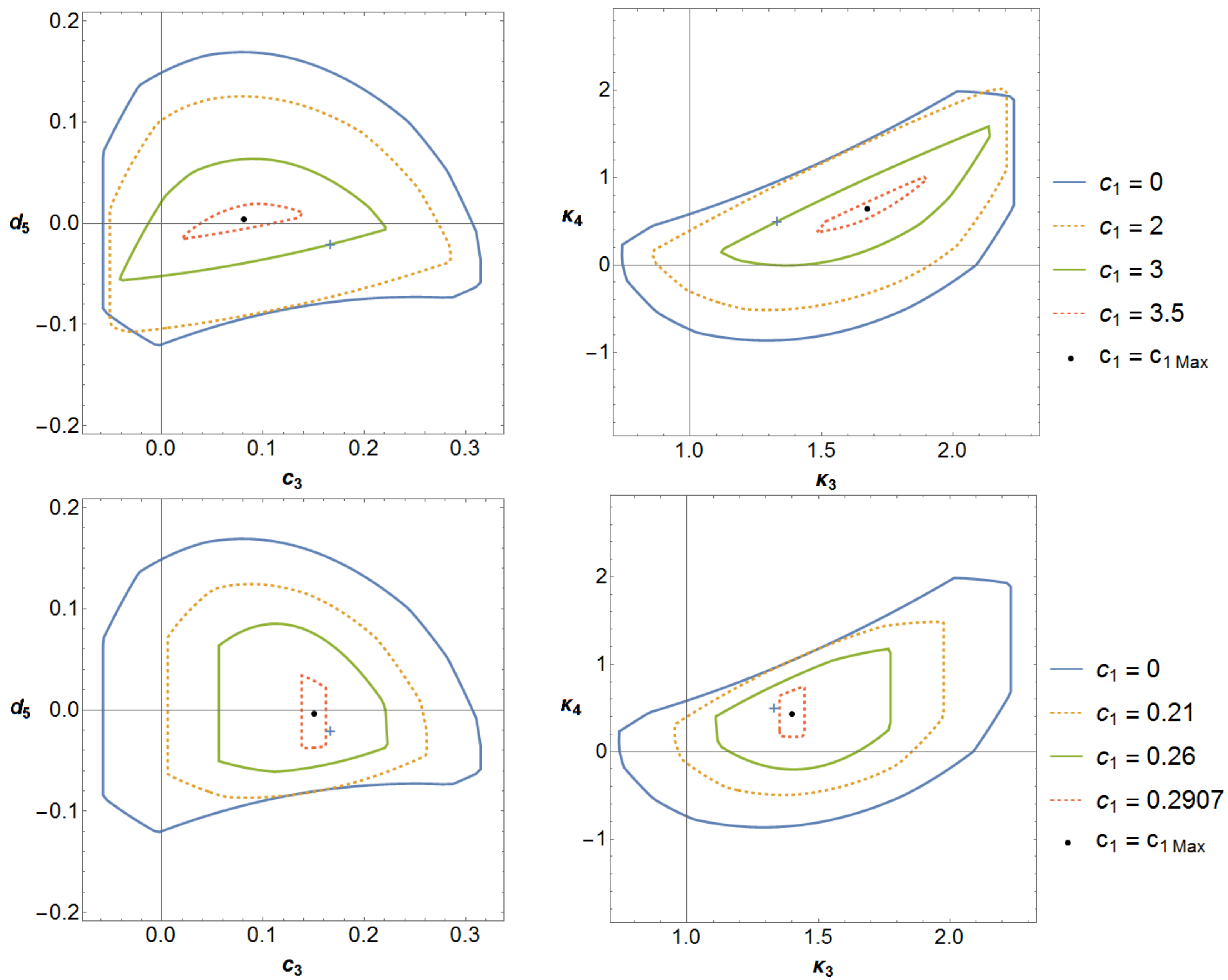
$$f_{\tau_1\tau_2}(v,t) = \frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}\mu\frac{\mathrm{Abs}_s\mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(\mu,t)}{(\mu-2m^2+t/2-v)^{N_S+1}} + \frac{1}{\pi}\int_{4m^2}^{\infty}\mathrm{d}\mu\frac{\mathrm{Abs}_u\mathcal{T}^{+}_{\tau_1\tau_2\tau_1\tau_2}(4m^2-t-\mu,t)}{(\mu-2m^2+t/2+v)^{N_S+1}}$$

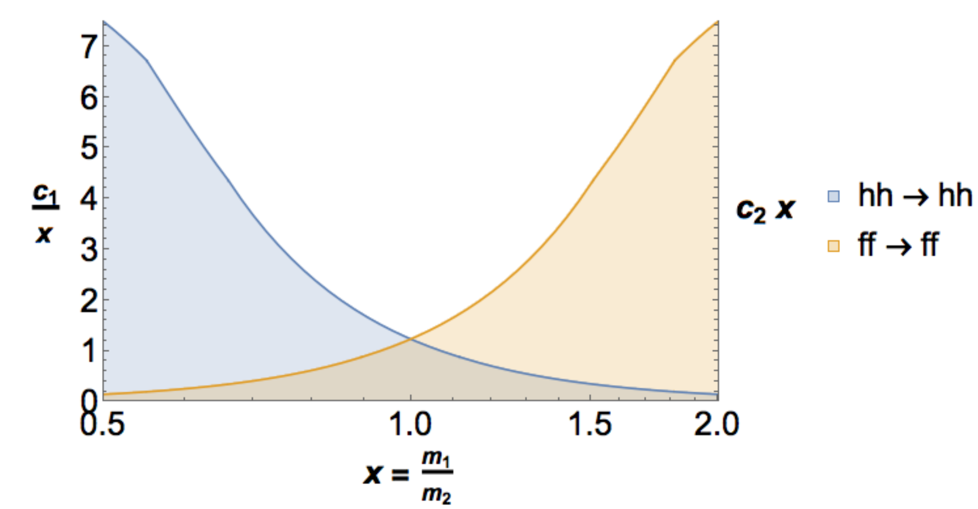
Scattering of spin states produces compact bounds!

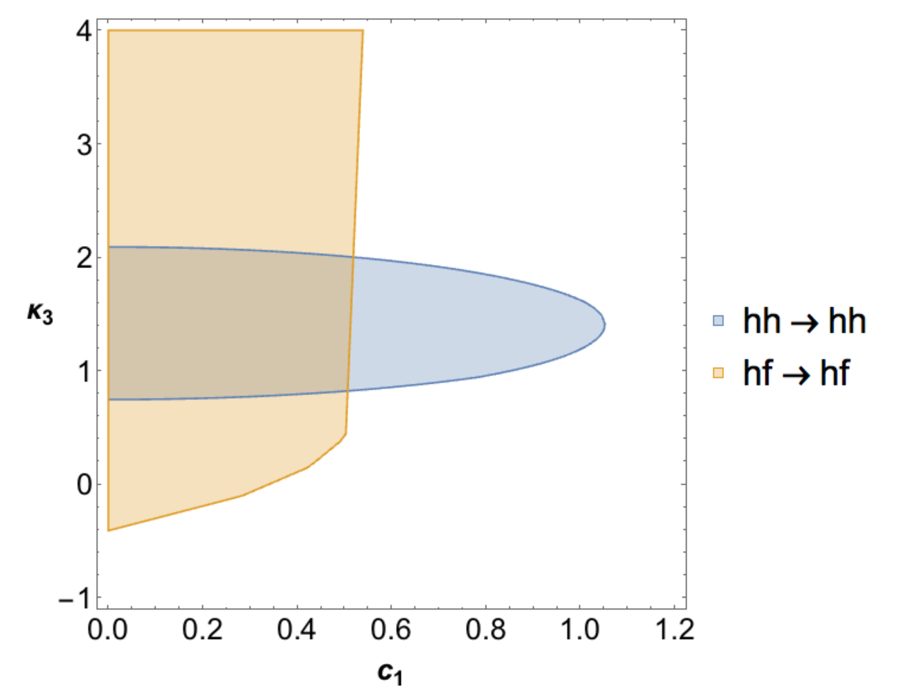
Cheung and Remmen 2016, Alberte, de Rham, Momeni, Rumbutis, AJT 2020

$$g^{(1)}_{\mu\nu} = (\eta_{\mu\nu} + h_{\mu\nu})^2\,, \qquad g^{(2)}_{\mu\nu} = (\eta_{\mu\nu} + f_{\mu\nu})^2$$

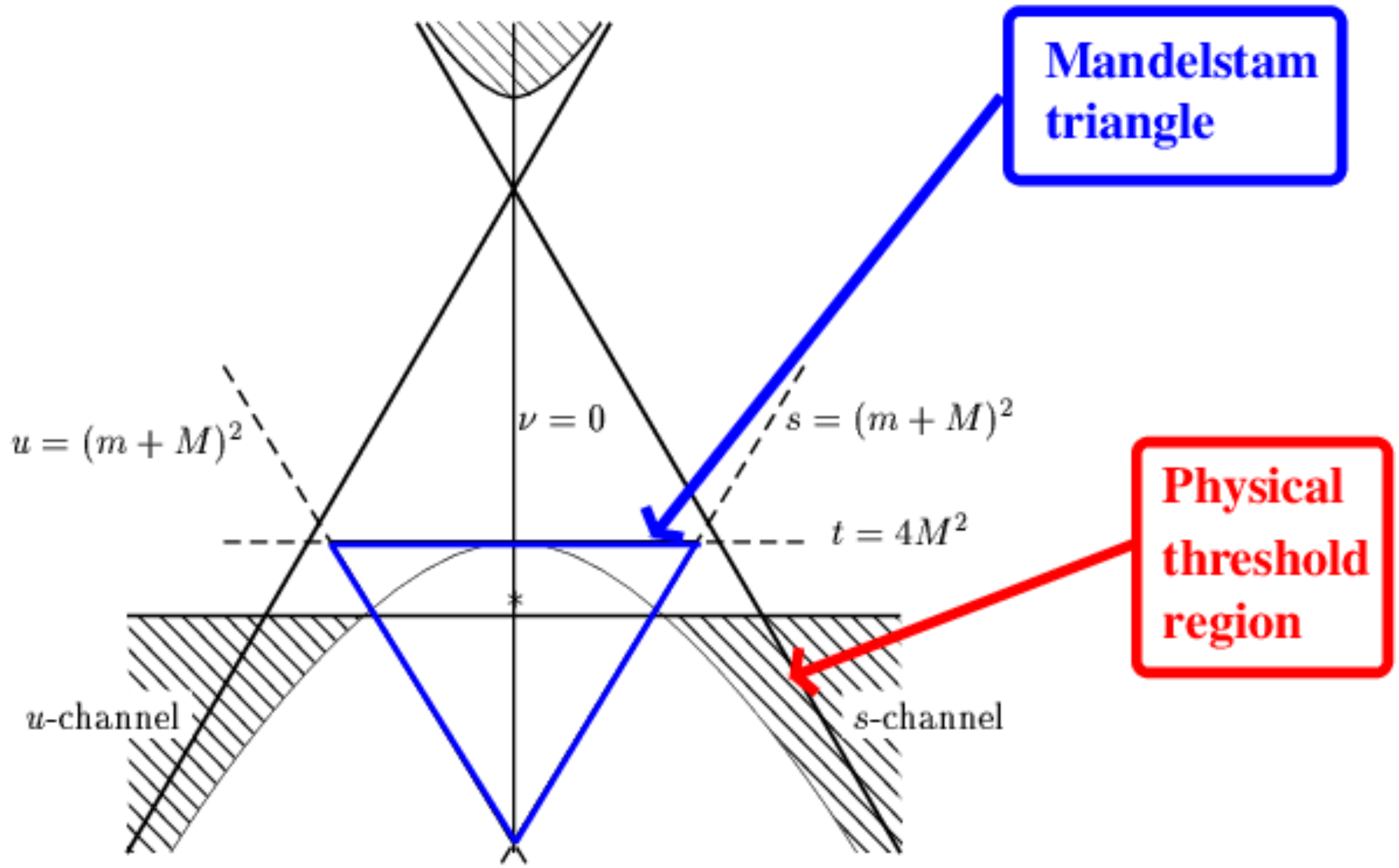
$$\mathcal{L}_{\rm int} = \frac{\gamma m^2 M_{Pl}^2}{2} c_1 \mathcal{L}_{hhf} + \frac{\gamma m^2 M_{Pl}^2}{2} c_2 \mathcal{L}_{hff} + \frac{\gamma m^2 M_{Pl}^2}{4} \lambda \mathcal{L}_{hhff}$$







Crossing Symmetry



$$A_s(s,t,u) = A_t(t,s,u)$$

$$a(t) + \int_{4m^2}^{\infty} \frac{\mathrm{d}\mu}{\pi(\mu-\mu_p)^2}\left[\frac{(s-\mu_p)^2}{\mu-s} + \frac{(u-\mu_p)^2}{\mu-u}\right]\mathrm{Im}A(\mu,t)$$
$$= a(s) + \int_{4m^2}^{\infty} \frac{\mathrm{d}\mu}{\pi(\mu-\mu_p)^2}\left[\frac{(t-\mu_p)^2}{\mu-t} + \frac{(u-\mu_p)^2}{\mu-u}\right]\mathrm{Im}A(\mu,s)$$

Partial Wave Expansion

Partial wave expansion:

$$A(s,t) = F(\alpha)\frac{s^{1/2}}{(s-4m^2)^\alpha}\sum_{\ell=0}^{\infty}(2\ell+2\alpha)C_\ell^{(\alpha)}(\cos\theta)a_\ell(s), \quad \alpha = \frac{D-3}{2}$$

Gegenbauer polynomials

Positive spectral Density

$$\rho_{\ell,\alpha}(\mu) = \frac{F(\alpha)}{(\mu-\mu_p)^3}\frac{\mu^{1/2}}{(\mu-4m^2)^\alpha}(2\ell+2\alpha)\mathrm{Im}a_\ell(\mu)C_\ell^{(\alpha)}(1)$$

$$\geq 0$$

Ratios of amplitude coefficients can be written in terms of moments, e.g.

Defining moments:

$$\langle\!\langle X(\mu,l)\rangle\!\rangle = \frac{\sum_\ell \int \mathrm{d}\mu \rho_{\ell,\alpha}(\mu) X(\mu,l)}{\sum_\ell \int \mathrm{d}\mu \rho_{\ell,\alpha}(\mu)}$$

Defining amplitude expansion coefficients:

$$f^{(2N,M)} \equiv \frac{1}{2(2N+2)!}\partial_t^M \partial_s^{2N+2}\mathcal{A}'(s,t)\Big|_{s,t\to 0}$$

Using dispersion relation and partial wave expansion we infer

$$\boxed{\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \frac{2(-3+D)\ell + 2\ell^2}{(D-2)\mu} \right\rangle\!\!\right\rangle}$$

Null-constraints

AJT, Wang, Zhou 2020

Caron-Huot, Van Duong 2020

$$0 = \mathcal{A}(s,t) - \mathcal{A}(t,s) = \sum_\ell \int d\mu \rho_{\ell,\alpha}(\mu) \left[\frac{2H_{D,\ell} st(s^2-t^2)}{(D-2)D\mu^2} + \ldots \right]$$

$$\sum_\ell \int \mathrm{d}\mu \; \rho_{\ell,\alpha}(\mu) \frac{H_{D,\ell}}{\mu^2} = 0 \qquad\qquad H_{D,\ell} = \ell(\ell+D-3)[4-5D-2(3-D)\ell+2\ell^2]$$

From Cauchy-Schwarz:

$$\left(\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle \right)^2 = \left\langle\!\!\left\langle \frac{2(D-3)\ell+2\ell^2}{(D-2)\mu} \right\rangle\!\!\right\rangle^2 \leq \left\langle\!\!\left\langle \left(\frac{2(D-3)\ell+2\ell^2}{(D-2)\mu} \right)^2 \right\rangle\!\!\right\rangle$$

ZERO!!!

BUT!!!

$$(2(D-3)\ell+2\ell^2)^2 = (5D-4)\left[2(D-3)\ell+2\ell^2\right] + 2H_{D,\ell}$$

hence:

$$\left(\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle \right)^2 \leq \frac{5D-4}{D-2} \left\langle\!\!\left\langle \frac{2(D-3)\ell+2\ell^2}{(D-2)\mu^2} \right\rangle\!\!\right\rangle$$

Two-sided bounds!!!

given:

$$\left\langle\!\!\left\langle \frac{2(D-3)\ell + 2\ell^2}{(D-2)\mu^2} \right\rangle\!\!\right\rangle < \frac{1}{\Lambda^2} \left\langle\!\!\left\langle \frac{2(D-3)\ell + 2\ell^2}{(D-2)\mu} \right\rangle\!\!\right\rangle$$

then:

$$\left(\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle \right)^2 < \frac{5D-4}{(D-2)\Lambda^2} \left(\frac{f^{(0,1)}}{f^{(0,0)}} + \left\langle\!\!\left\langle \frac{3}{2\mu} \right\rangle\!\!\right\rangle \right)$$

$$-\frac{3}{2\Lambda^2} < \frac{f^{(0,1)}}{f^{(0,0)}} < \frac{5D-4}{(D-2)\Lambda^2}$$

Weakly Broken Galileon

AJT, Wang, Zhou 2020

$$\mathcal{A}'(s,t) \sim \frac{1}{\Lambda_3^{D-4}} \left(\frac{m^2}{\Lambda_3^6} x + \frac{1}{\Lambda_3^6} y + \frac{1}{\Lambda_3^8} x^2 + \dots \right)$$

$$y = stu$$

$$x = s^2 + t^2 + u^2$$

not suppressed

suppressed

$$-\frac{3}{2\Lambda^2} < \frac{f^{(0,1)}}{f^{(0,0)}} < \frac{5D-4}{(D-2)\Lambda^2}$$

Contradiction!!!

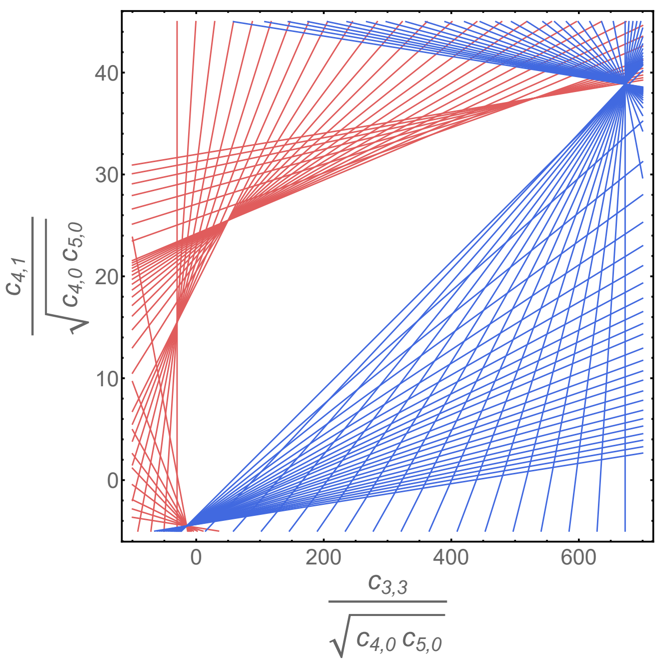
No local UV completion for weakly broken Galileons

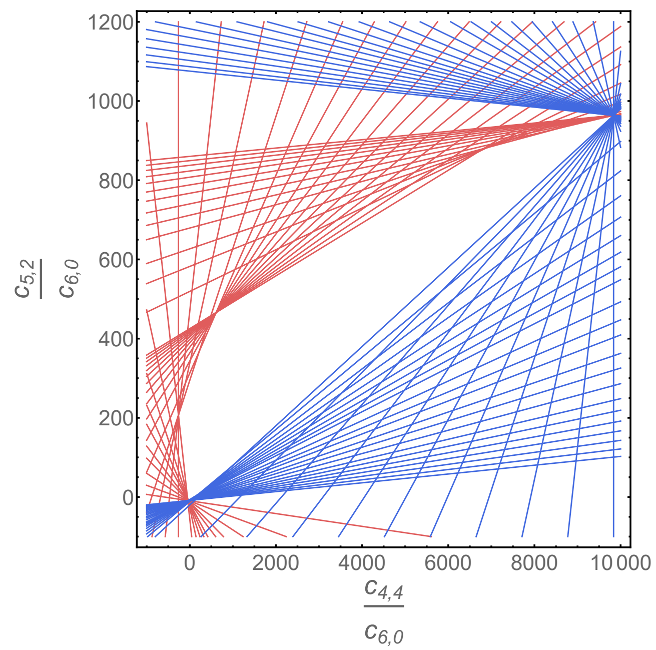
see also Bellazzini et al, Positive Moments .., 2020

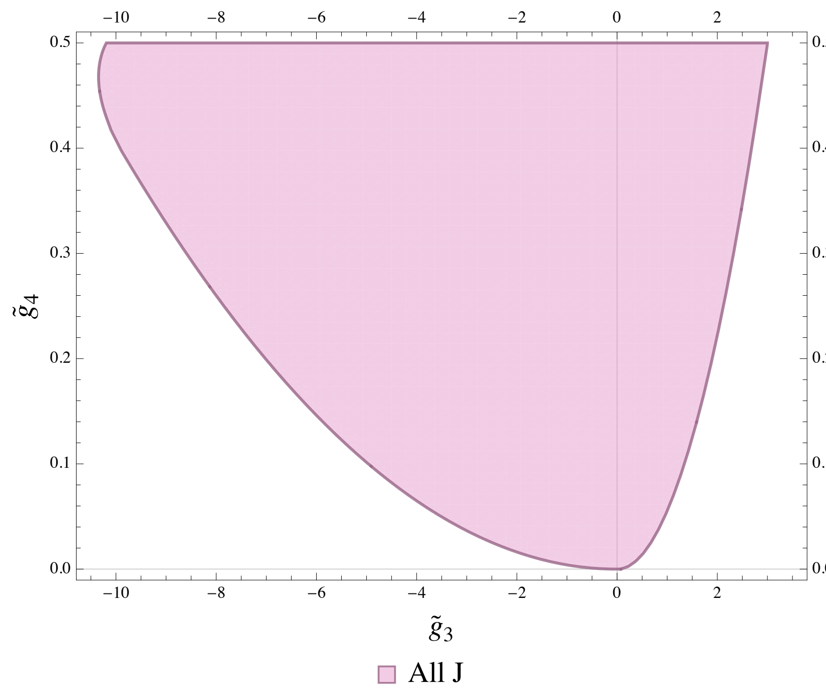
Compact positivity bounds

$$\mathcal{A}'(s,t) = \sum_{p.q=0}^{\infty} c_{p,q} w^p t^q \qquad w = -(s-2m^2)(u-2m^2)$$

AJT, Wang, Zhou 2020







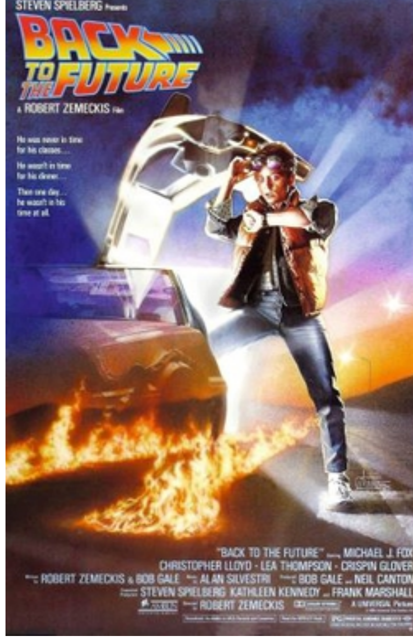
Caron-Huot, Van Duong 2020

$$\mathcal{A}'(s,t,u) = g_2(s^2+t^2+u^2) + g_3(stu) + g_4(s^2+t^2+u^2)^2 + g_5(s^2+t^2+u^2)(stu) + \cdots$$

+ Much recent numerical work: Rewrite as a linear optimisation problem (extremize Wilson coefficients subject to null constraints) or variant

Compact positivity bounds and causality

Carrillo Gonzalez, de Rham, Pozsgay, AJT ‘Causal Effective Field Theories’ 2023

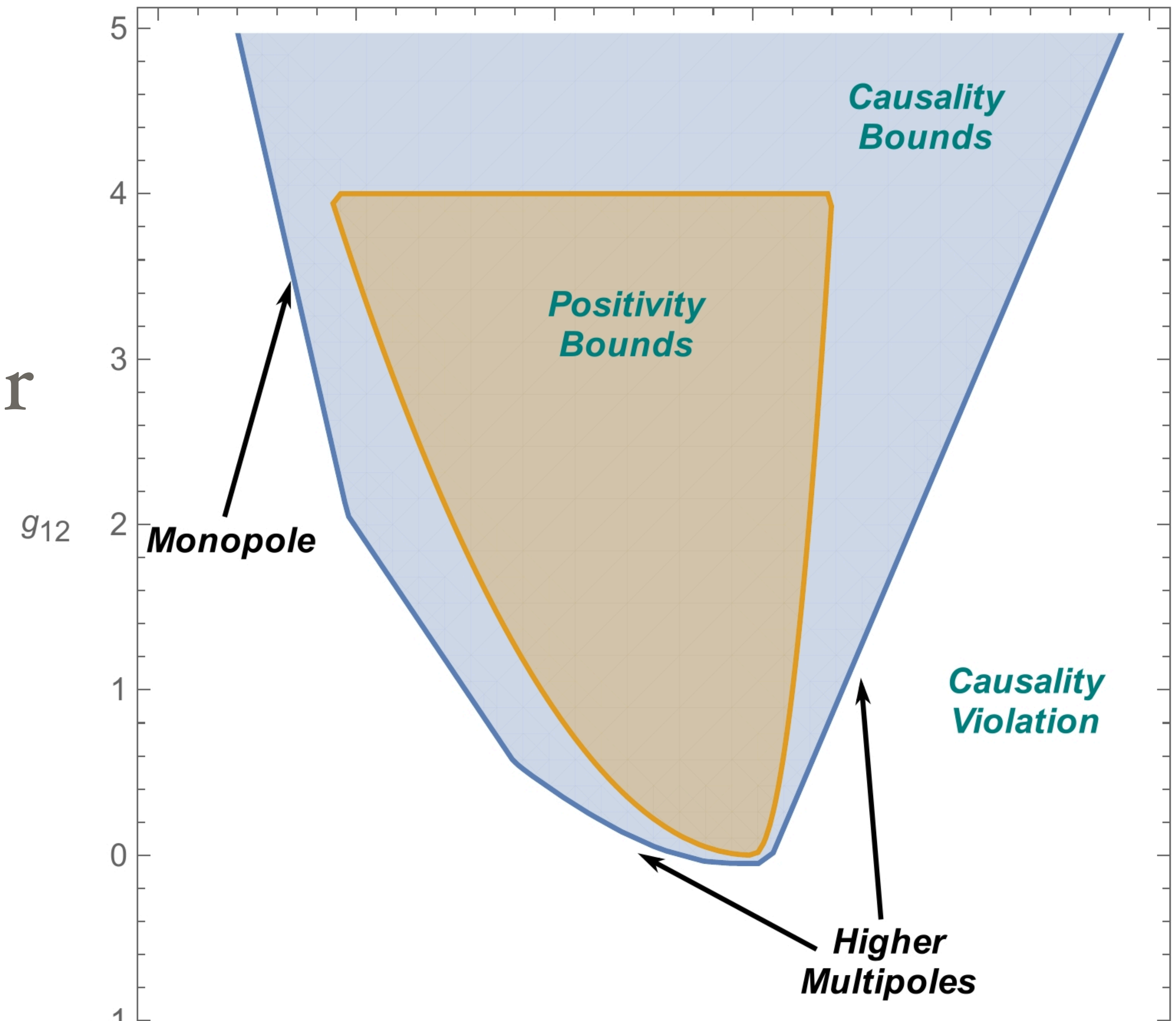


For Goldstone model:

$$\mathcal{L} = -\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 + \frac{g_8}{\Lambda^4}(\partial\phi)^4 + \frac{g_{10}}{\Lambda^6}(\partial\phi)^2\left[(\phi_{,\mu\nu})^2 - (\Box\phi)^2\right] + \frac{g_{12}}{\Lambda^8}((\phi_{,\mu\nu})^2)^2 - g_{\text{matter}}\phi J$$

Causality =
positivity of Eisenbud-Wigner
scattering time delay

$$\Delta T_\ell = 2\frac{\partial \delta_\ell}{\partial \omega}\bigg|_\ell \quad \gtrsim -\omega^{-1}$$



Including massless gravity

Caron-Huot et al, Sharp Boundaries for the Swampland, 2102.08951

Key idea

Construct quantity which is not obviously positive, but is well defined for $t < 0$ such that the two subtraction dispersion relation can be used

Find functions $f(p)$ such that where $t = -p^2$

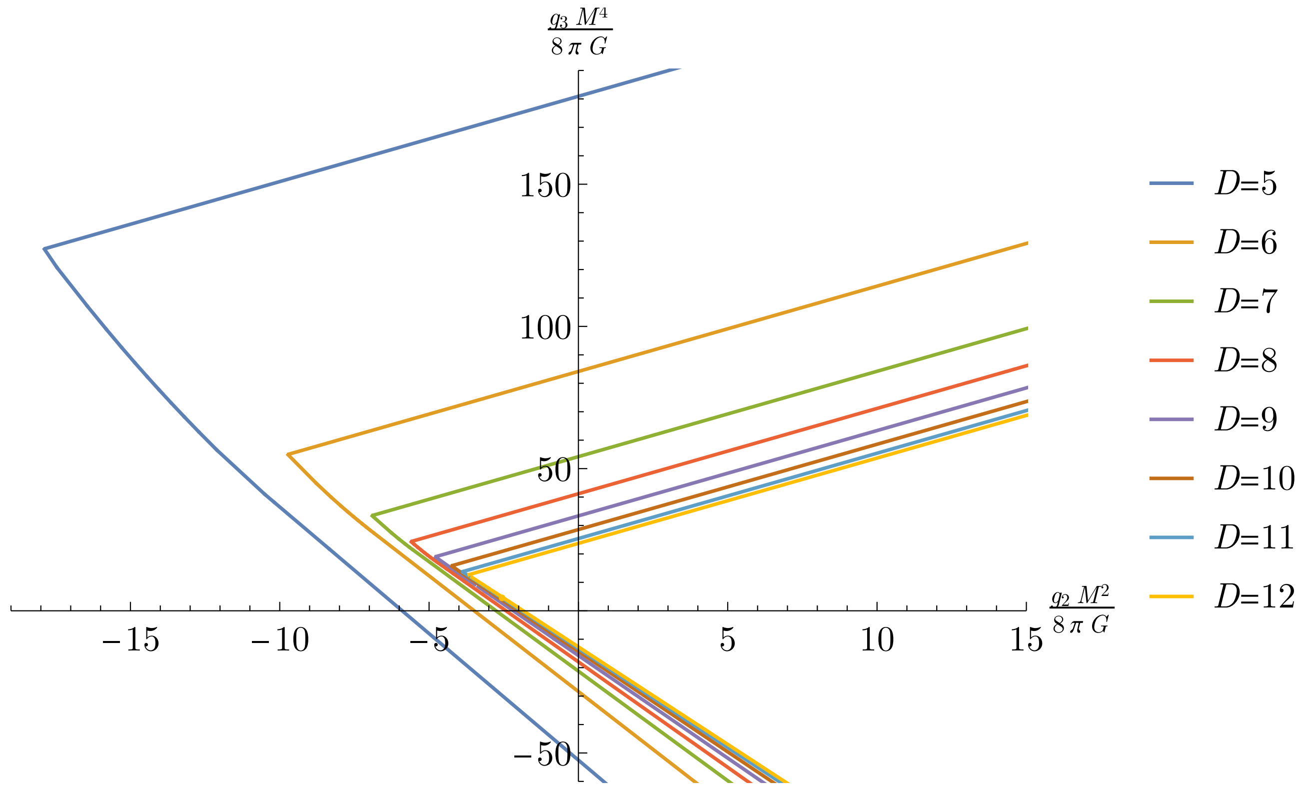
$$\int_0^M dp f(p) \left[\frac{1}{2}\partial_s^2 A(s,t) + \dots\right] > 0$$

$$\Downarrow$$

$$\int_0^M dp f(p) \left[\frac{1}{M_{\rm Pl}^2 p^2} + \frac{\tilde{c}}{M^4} \dots\right] > 0 \rightarrow \quad \tilde{c} > -\frac{M^2}{M_{\rm Pl}^2} \frac{\int_0^M f(p) \frac{M^2}{p^2}}{\int_0^M f(p)}$$

Positivity of Goldstones coupled to Gravity

Caron-Huot et al, Sharp Boundaries for the Swampland, 2102.08951



Caron-Huot et al, Sharp Boundaries for the Swampland, 2021

In 4D

$$\tilde{c} > -\frac{M^2}{M_{\mathrm{Pl}}^2} 17 \log(1.7 M b_{\mathrm{max}})$$

Previously conjectured form in

Positivity Bounds and the Massless Spin-2 Pole
Alberte, de Rham, Jaitly, AJT 2000

Positivity of EFT of Gravity

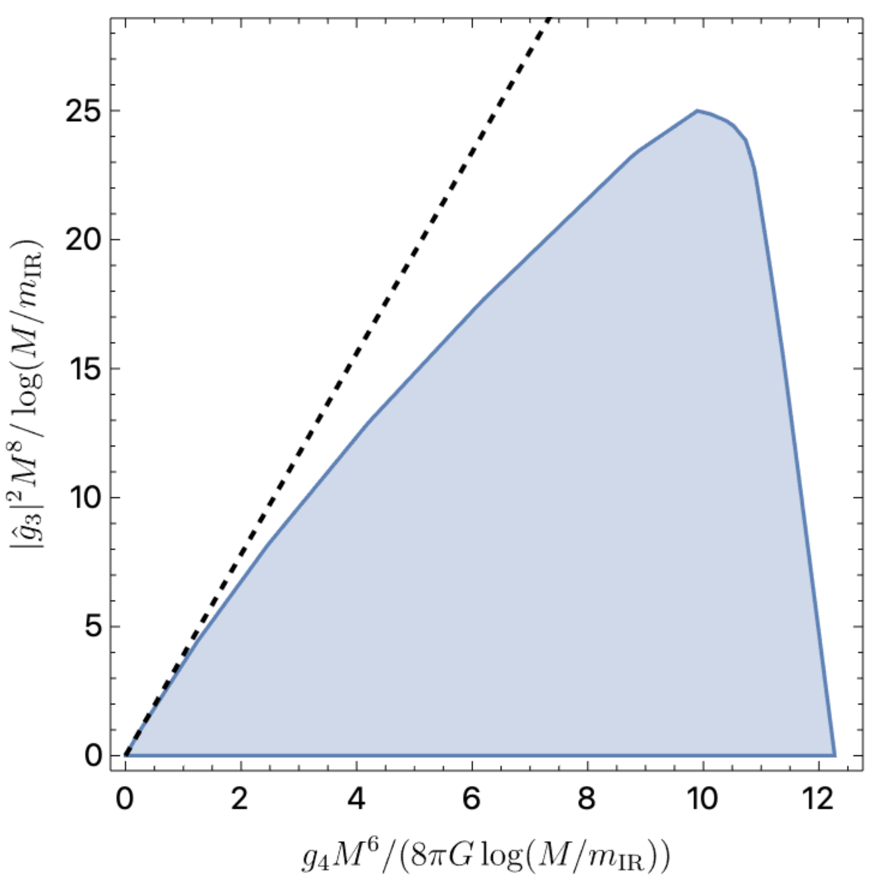
'Causality constraints on corrections to Einstein gravity'
Caron-Huot, Li, Parra-Martinez, Simmons-Duffin 2022
'Graviton partial waves and causality in higher dimensions'
Caron-Huot, Li, Parra-Martinez, Simmons-Duffin 2022
'Crossing Symmetric Spinning S-matrix Bootstrap: EFT bounds'
Chowdhury, Ghosh, Holder, Raman, Sinha 2022
'Constraints on Regge behaviour from IR physics'
de Rham, Jailty, AJT 2023

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\Big[R - \frac{1}{3!}\left(\alpha_3 R^{(3)} + \tilde{\alpha}_3 \tilde{R}^{(3)}\right) + \frac{1}{4}\left(\alpha_4 (R^{(2)})^2 + \alpha_4' (\tilde{R}^{(2)})^2 + 2\tilde{\alpha}_4 R^{(2)}\tilde{R}^{(2)}\right) + \dots\Big] + S_{\rm matter}$$

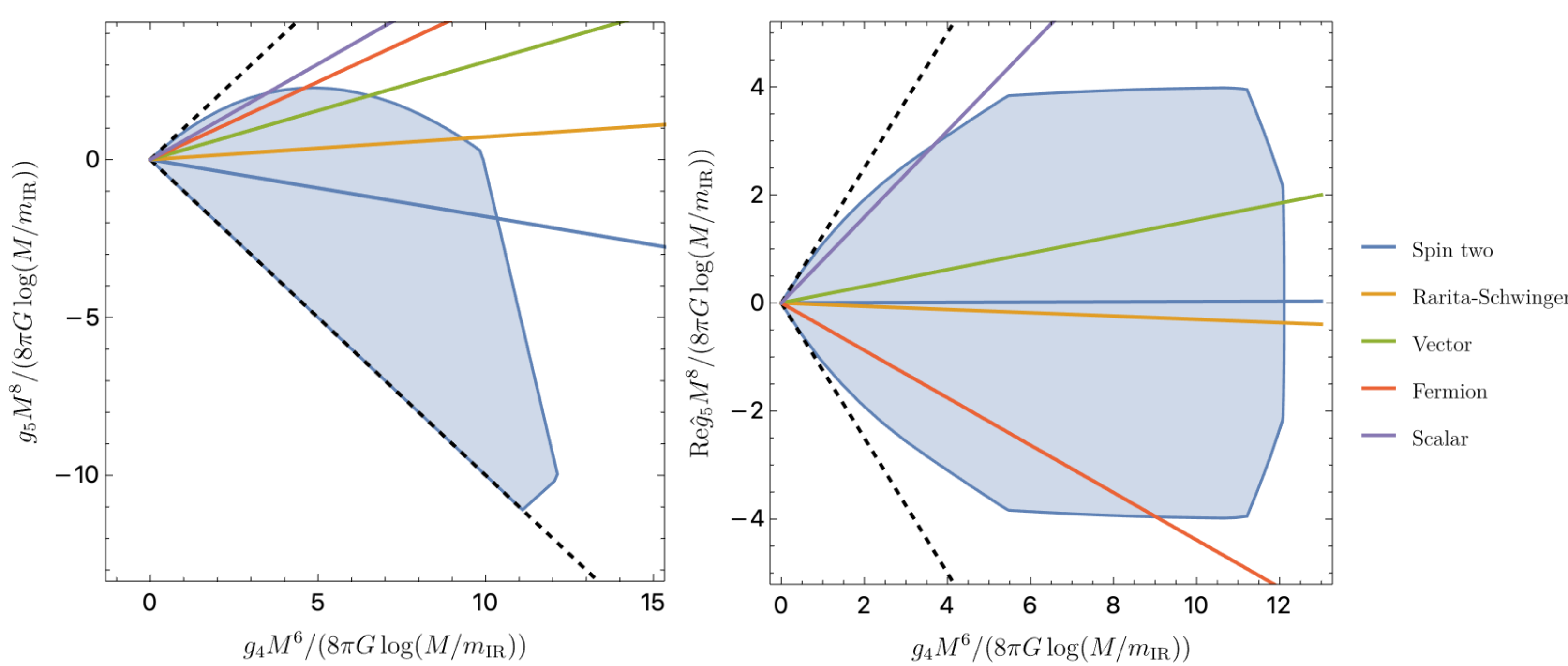
$$R^{(2)} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\,, \qquad \tilde{R}^{(2)} = R_{\mu\nu\rho\sigma}\tilde{R}^{\mu\nu\rho\sigma}\,, \qquad \tilde{R}_{\mu\nu\rho\sigma} \equiv \tfrac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta}R_{\alpha\beta\rho\sigma}\,,$$

$$R^{(3)} = R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}R_{\alpha\beta}{}^{\mu\nu}\,, \qquad \tilde{R}^{(3)} = R_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{\alpha\beta}\tilde{R}_{\alpha\beta}{}^{\mu\nu}\,.$$

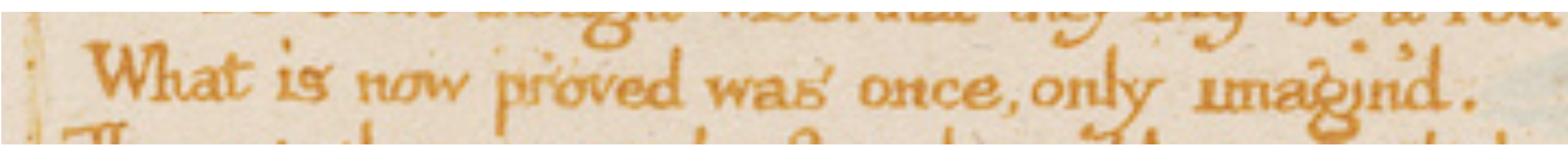
$$\widehat{g}_3 = \alpha_3 + i\tilde{\alpha}_3, \qquad g_4 = 8\pi G(\alpha_4 + \alpha_4')\,, \qquad \widehat{g}_4 = 8\pi G(\alpha_4 - \alpha_4' + i\tilde{\alpha}_4)$$



Caron-Huot, Li, Parra-Martinez, Simmons-Duffin 2022



- Positivity Bounds are very powerful at constraining irrelevant operators in a low energy EFT
- Full crossing symmetry implies upper and lower bounds on Wilson coefficients
- Strong constraints on interacting massive spin theories and supersoft theories
- With some assumptions can be applied to gravitational effective theories massless gravity
- Results broadly consistent with expectations of naive EFT counting/naturalness arguments



William Blake, from 'The Marriage of Heaven and Hell'